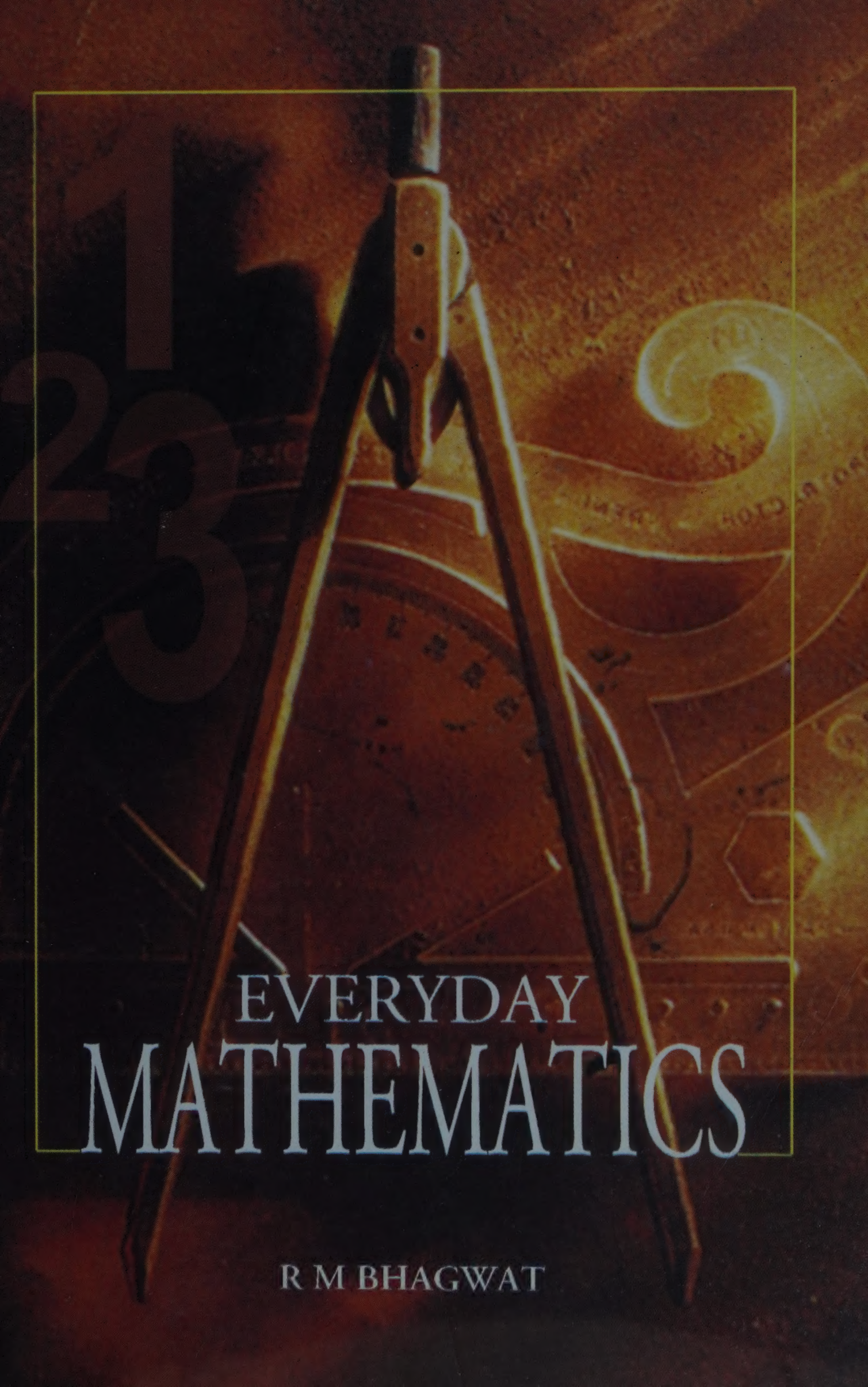
EVERYDAY
MATHEMATICS
R M BHAGWAT

EVERYDAY MATHEMATICS

Popular Science

EVERYDAY MATHEMATICS

R.M. BHAGWAT

Illustrator
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To
my father and mother

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Mumbai

R.M. BHAGWAT

Preface

Mathematics is a creation of the human mind originating from human activities and observation of Nature. Mathematics goes to the very basic thinking processes of the human mind to express them in precise terms. It converts the world of reality into a world of concepts and studies the laws of abstract concepts which govern the real world. Most of everyday mathematics is the essence of these basic concepts and therefore simple to fathom. Although many concepts occur to our mind intuitively, it is necessary to build up a suitable vocabulary and certain rules and symbols to express them precisely and briefly. Thus, mathematics has its own language and script, which has to be learnt first. This is perhaps the reason why it is thought to be unrelated to everyday life and hence, difficult to comprehend; while in reality, it is not only intimately connected with real life but originates from it. Ironically, most people detest mathematics and turn their backs to it, while, in fact, it is of use in all walks of life and for all other branches of knowledge. It is not a mere coincidence that great scientists like Archimedes, Newton, Gauss and Lagrange have made great contributions to mathematics as well as to science.

Mathematics is perhaps one of the first disciplines of human knowledge and is as old as the civilisation itself. It has grown in volume and in complexity with the volume and complexity of human life. Life of man has changed gradually throughout the history of civilisation, from the simple life of a cave-man to an extremely complex and multifaceted life of modern man. With this, mathematics has also grown into a vast and rich branch of

human knowledge. Although the common man need not go much beyond the mathematics of, say, a thousand years ago, scientists, mathematicians, technologists, economists and others make use of the most sophisticated systems in mathematics in everyday life in one way or the other, on an unprecedented scale. Mathematics is now an all-pervasive element of everyday life.

It is, therefore, necessary that people from all walks of life should understand and appreciate the contribution of mathematics to the progress of mankind. The book explains the nature of everyday mathematics in simple language with a number of illustrations drawn from our life. The historical development of mathematics has been described, emphasising how mathematics originates from simple everyday activities and how it helps solve problems in daily life. It is a matter of pride for us that throughout the development of mathematics up to the twelfth century, almost all important steps in its progress were based upon the findings of Indian mathematicians. The contributions of ancient Hindu mathematicians has been duly emphasised in this book. That mathematics is not a mere collection of abstract concepts and rules but is the very basis of everyday life has been explained with the help of a number of everyday problems. Essential aspects of mathematics have been explained in simple language, using the minimum of mathematical terms and formulae. The book has been amply illustrated with a number of diagrams. Every effort has been made to make the book as interesting and as readable as possible.

It is hoped that *Everyday Mathematics* will serve to encourage readers to understand, appreciate and study mathematics.

1

INTRODUCTION

Since the earliest times mathematics has been one of the most useful of subjects. The word 'mathematics' comes from the Greek *mathemata*, meaning 'things that are learned'. According to Bertrand Russel, the famous twentieth-century British mathematician and philosopher, "Mathematics may be defined as the subject in which we never know what we are talking about, nor whether what we are saying is true." It tells us nothing about the real world, but states that one thing follows from another in a certain way. It may seem strange to apply this phrase to a single field of knowledge, but for the Greeks, mathematics included not only the study of numbers and space but also astronomy and music as mathematical subjects; yet, the scope of mathematics today is broader than ever before.

The origins of mathematics are lost in antiquity but we know that it was used 4,000 years ago by ancient peoples, such as the Babylonians and Egyptians, to work out the calendar, so that they could know in advance when to sow their crops, or when the river Nile was going to flood, or to solve quadratic equations. They even knew about the theorem now wrongly attributed to Pythagoras. Their cultures were based on farming, and they needed methods of surveying and plotting the courses of stars and planets quite accurately. Arithmetic was used in trade for exchange of money and goods and for keeping accounts. Geometry was used for marking out boundaries

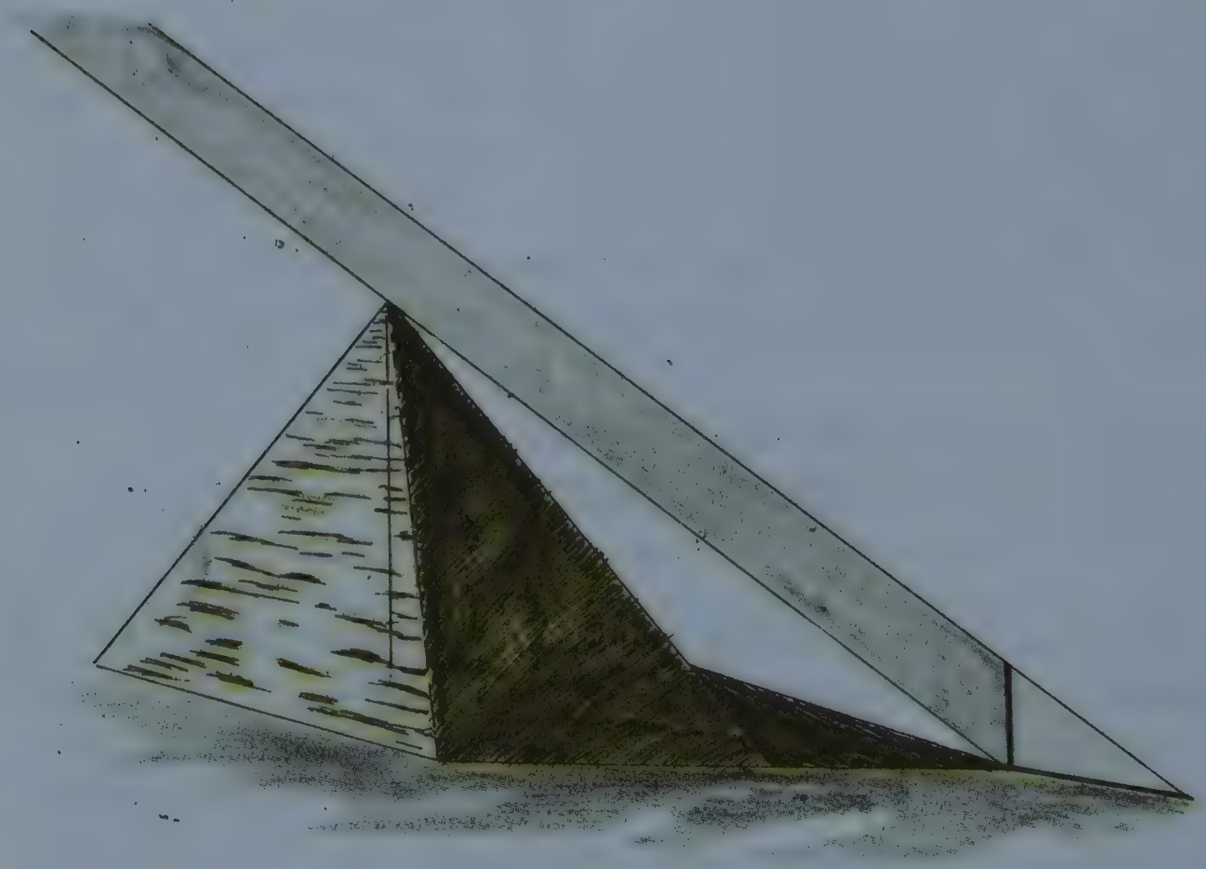

Fig. 1

of fields and for building monuments such as the pyramids.

The first theoretical mathematician is believed to be Thales of Miletus (645-546 B.C.). He showed that it was possible to calculate the height of an object by measuring its shadow and comparing it with the shadow cast by a measuring stick (Fig.1). He is supposed to have predicted an eclipse of the sun. His pupil, Pythagoras, established geometry as a recognised science among the Greeks and prepared the way for Euclid and Archimedes.

The Greeks added much to what they had inherited from the Babylonians. In addition, they made mathematics into a logical system, which begins with some basic ideas that are assumed to be true (called axioms) and then goes on to make conclusions (called proofs).

All of us are mathematicians to some extent. We use mathematics every day in our lives—when we consult a watch or clock to find out time, when we calculate the cost or purchases and the change due to us, when we play sports like football, tennis or cricket and keep the score.

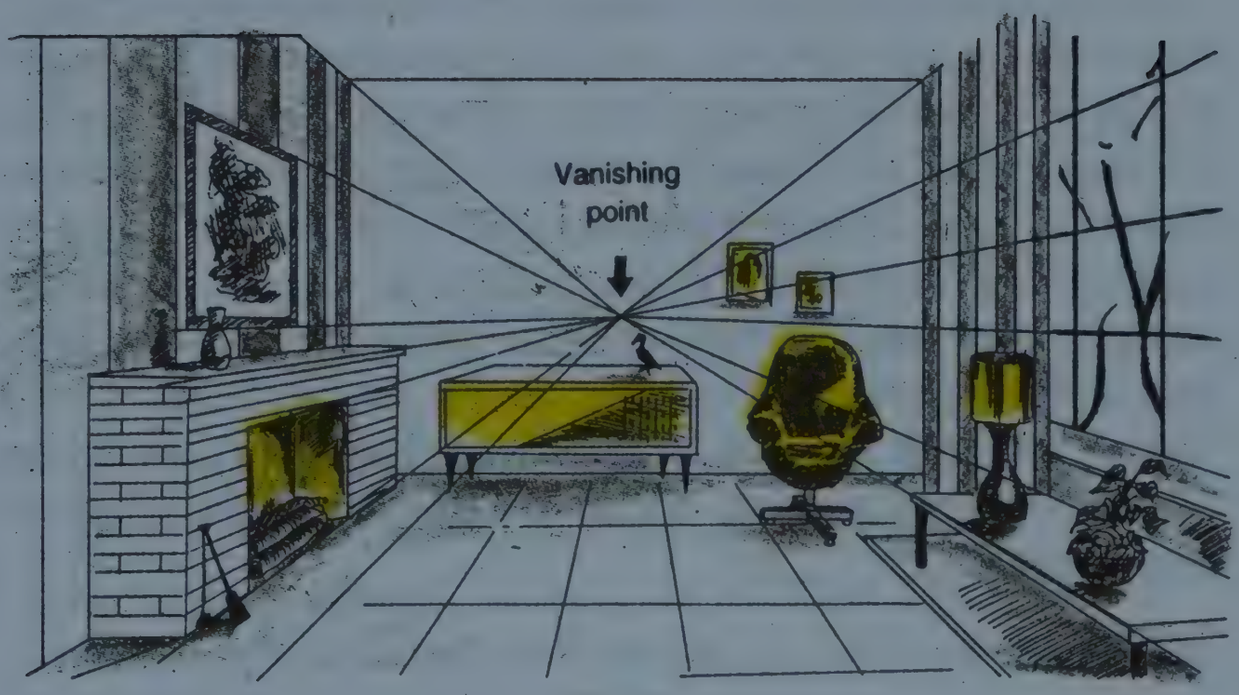

Fig. 2

The accounting operations of business and industry are based on mathematics. Insurance is largely a matter of the compounding of interest. The pilot of a ship or plane uses geometry in piloting his course. Much of the surveying is based on trigonometry. Mathematics helps even the artist in drawing, let's say, the perspective—the system by which the artist represents on a flat surface objects and persons as they actually appear in a three-dimensional world (Fig. 2). In music, scales and theories of harmony and counterpoint are based on mathematics. Mathematics is so important in science and serves so many of its branches that it has been called the 'queen and servant of science', by mathematician Eric Temple Bell. Measurements and other mathematical techniques are vital for the physicist. Chemists use logarithms to calculate the acidity of a substance—the pH value. Astronomers measure angles and areas to study the motion of the sun, stars, the moon and planets. In biology dimensional analysis is used to analyse the growth pattern of certain animals.

With high-speed electronic computers, calculations can be performed in a fraction of a time in which people

could do them using other methods, so their invention has revolutionised the type of things for which mathematics can be used. As calculations involving astronomical measurements and time keeping became more accurate, navigation became easier, and from the time of Christopher Columbus onwards men ventured further and further from home to explore new territories, making maps as they went. Mathematics was used to design better ships, railway engines, cars and eventually aeroplanes, and it was also used in the design of radar systems, and to send rockets to the moon and some of the planets. Such then, in brief, is the scope of mathematics.

2

THE NUMBERS

Numbers Everywhere

Our life is governed by numbers. Consider the following situations in a common man's life:

- In the early morning, the sound of an alarm clock awakens an office-goer. *"It's 6 o'clock; must get up."* Thus starts his day.
- In a bus, a conductor to a passenger: *"Give me 40 paise more."*
  Passenger: *"Why, I have given you the correct fare."*
  Conductor: *"There is a 25 per cent increase now."*
  Passenger: *"Oh!"*
- A housewife, at a milk booth in a city: *"Give me a two-litre bag."*
  "I don't have a two-litre bag."
  "Okay, give me a one-litre bag and two half-litre bags."
- In a restaurant, a customer looking at a bill, says: *"Waiter, you have not added the bill properly: it should be Rs 8.50 instead of 9.50."*
  "Sorry, sir!"

These are a few instances where numbers are used in our everyday life. Numbers are also crucial in not so common fields of life. A difference of 0.001 second in a sprinter's time may make him earn or lose a gold medal. A difference of even one-thousandth of a centimetre in the diameter of a wheel may make it useless for a watch.

A telephone number, a number on a ration card, a number of a bank account or an examination number helps us in identifying a person.

How old are these numbers? Who invented them? What is the origin of numbers? These are some of the questions that naturally come to our mind. The answers to these questions are as interesting as the questions themselves. Let us try to learn a few facts about numbers.

The Origin of Numbers

Numbers are almost as old as the human civilisation itself. In the Ashmolean Museum at Oxford, there is an Egyptian royal mace on which there is a record of 120,000 prisoners and of booty consisting of 400,000 oxen and 1,422,000 goats. The record dating back to before the year 3400 B.C. shows that in antiquity people knew how to write large numbers. Of course, the beginnings of the use of numbers must go back long before the Egyptians.

The primitive man did not have much to do with

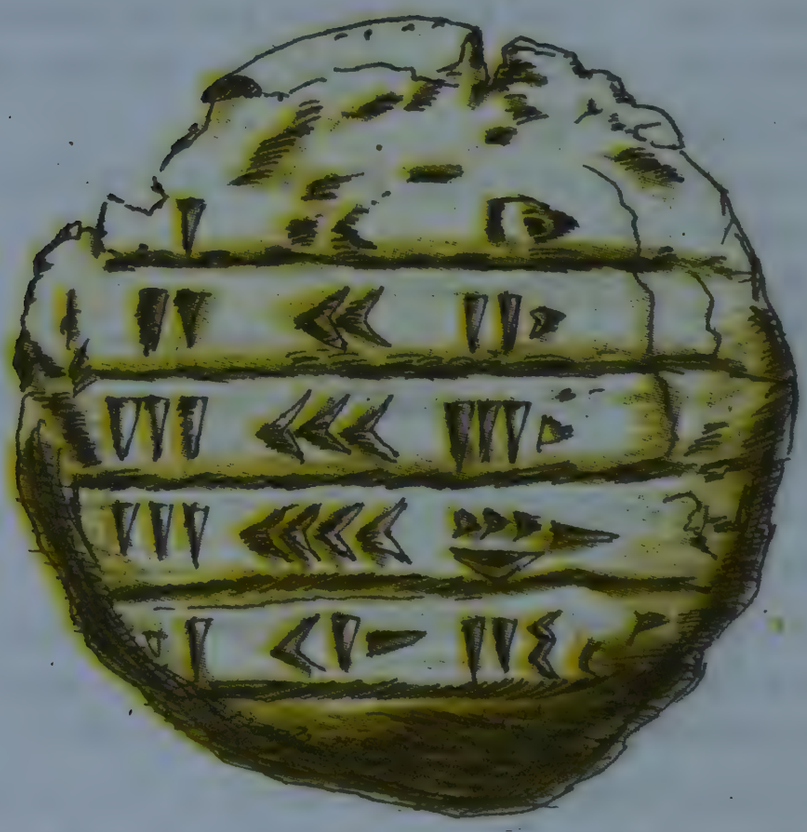

Fig. 3

counting: home was a cave at hand, food grew on trees or plants or was hunted with weapons. However, 10,000 years ago when primitive people began settling down into villages, started agriculture and began keeping animals, life became more complex than before. They felt the need to regulate their daily routine, their community life and the family life too. Numbers were needed to count their cattle, to compare their agriculture produce, to measure the land and to keep track of time. There were a number of civilisations in different parts of the world, such as Babylonia, Egypt, India, China and others. All these civilisations had developed their own primitive numbers, perhaps, at about the same time. Ancient clay statutes of the Babylonians are found with numbers written on them (Fig. 3). They developed cuneiform or wedge-shaped writing in which different symbols were produced with sharp-edged sticks on wet clay, formed into clay bricks. There were special symbols for one (𒁹), ten (𒌋), hundred (𒁹𒌍), etc. For writing a number the Babylonians repeated the symbols, e.g. to write 1,000

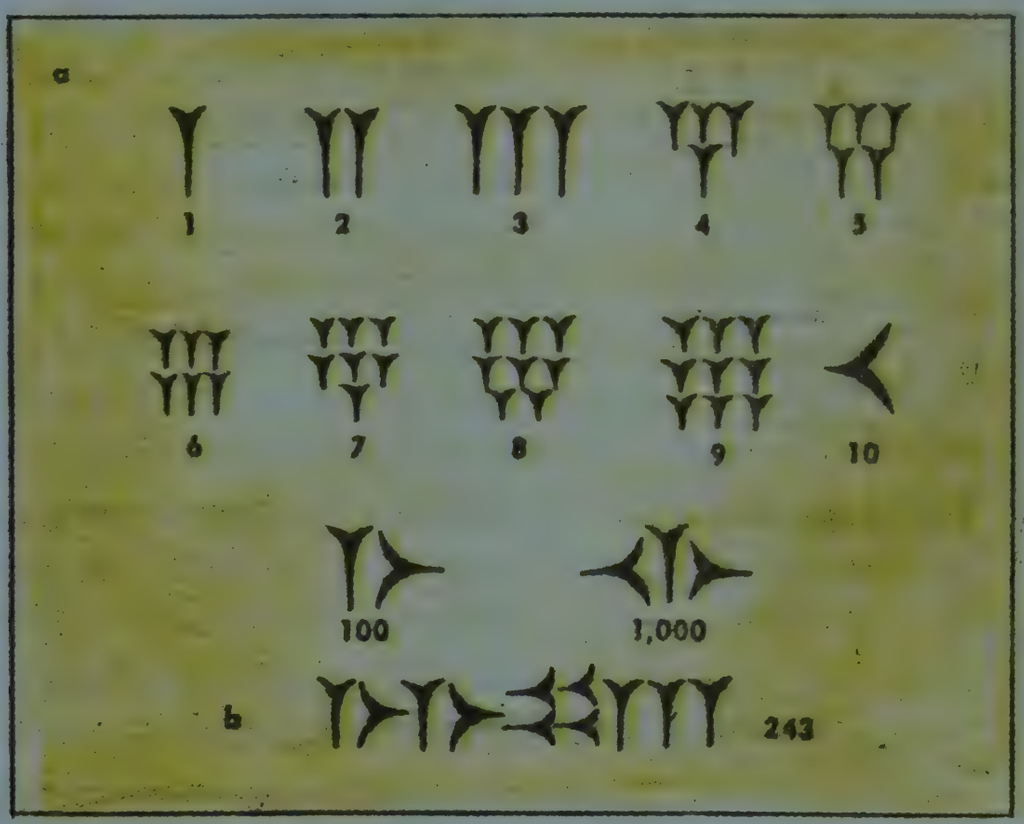

Fig. 4

Fig. 5

they used ≺𓎆≻ or repeated 100 ten times (Fig. 4). The Babylonians could count quite large numbers. They counted numbers from 1 to 60 and counted larger numbers in groups of 60, as we do at present with groups of 10. The ancient people in Egypt were also able to count large numbers and had found out that there were 365 days in a year. They used the hieroglyphic system of writing numbers on monuments, known as obelisks (Fig. 5). This system, however, was rather cumbersome when it came to writing large numbers (Fig. 6.a). Fig. 6.b. shows how the number 527 could be written.

These people used only a few elementary counting numbers in the beginning. Many of them could not count beyond 20, and some only upto 10 or 5. Even today many people in villages can count only up to 20 and some primitive tribes can count only up to 4. Little children learning to count find it difficult to count beyond 10.

At first, a person counted the number of animals in a herd by placing a

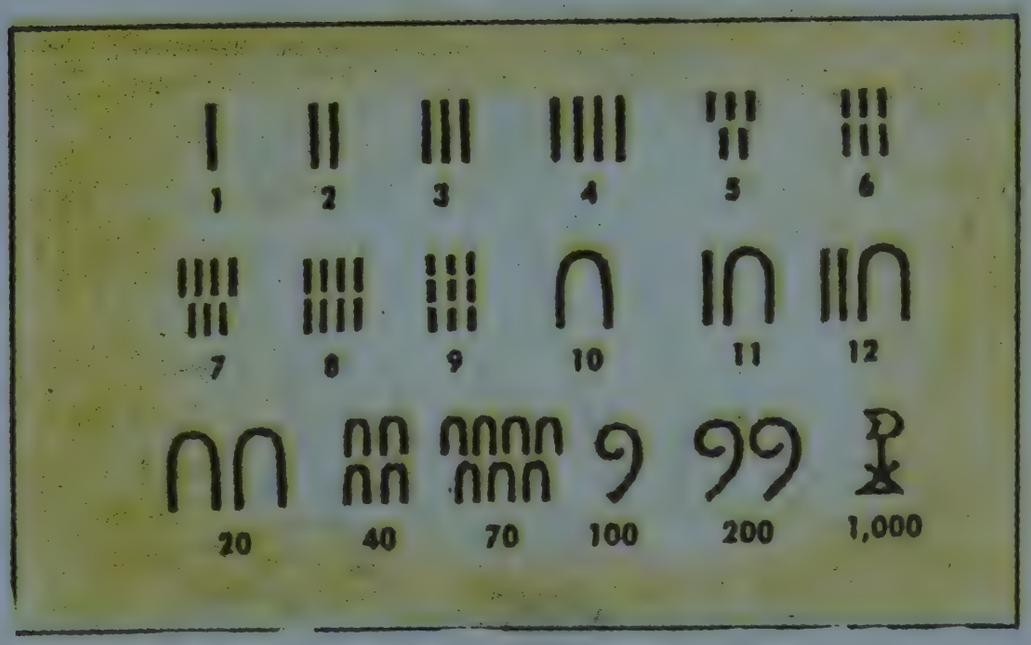

Fig. 6.a

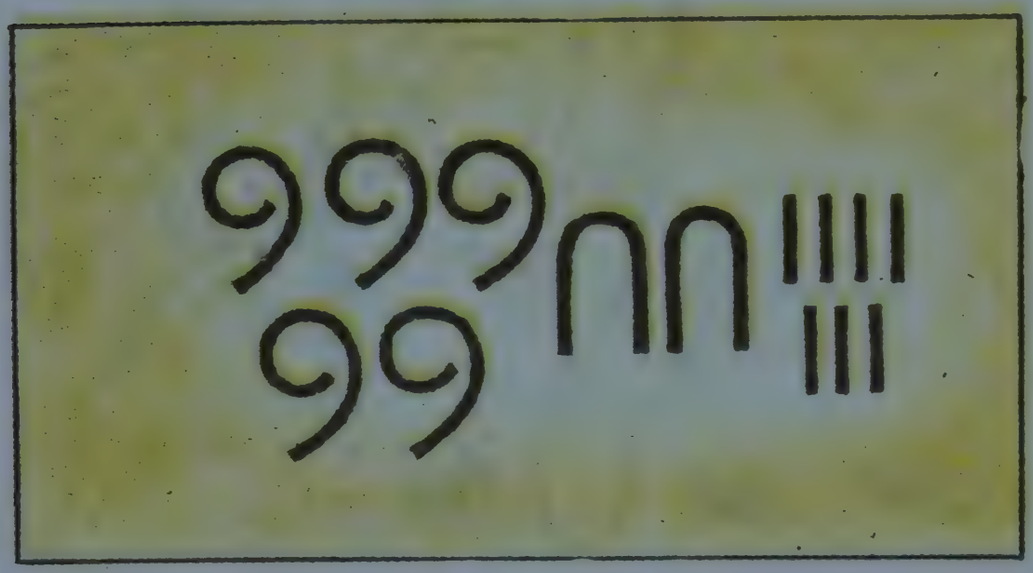

Fig. 6.b

pebble on the ground or tying a knot in a rope for each animal (Fig. 7). Each pebble in the growing heap or each knot in the rope would stand for a single animal. Later, probably man used his ten fingers for his calculation. We may surmise that when ten fingers had been counted, a little stone would be set aside to represent this first 10; the fingers would then be used to count another 10. Another stone would be added to the first one and so on. When the stones in the pile would equal the number of fingers, they would represent 10 tens (Fig. 8). The pile of 10 stones would then be removed to be replaced by a larger stone to indicate

Fig. 7

'10 tens' or a 100. Thus three large stones, seven small stones, eight sticks standing for eight fingers, would represent three hundreds, seven tens and eight units, i.e. the number 378 (Fig. 9). Not all primitive peoples would use 10 or the number of fingers on one's hands; some would use only the two hands and not the fingers; some may have used the toes too, and so on.

Since stones, pebbles and sticks are awkward to handle, people created symbols to represent numbers as soon as they learned to write.

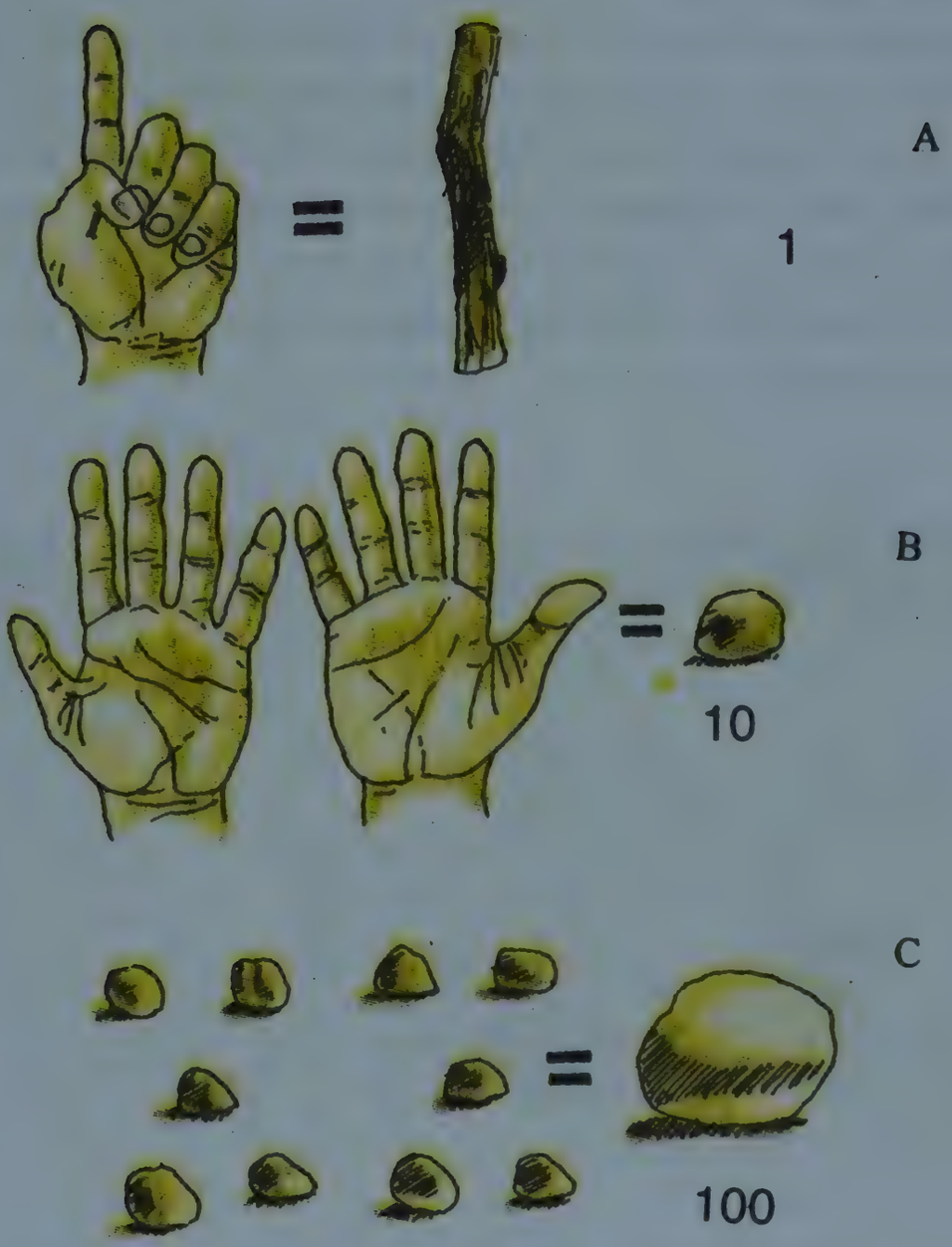

Fig. 8

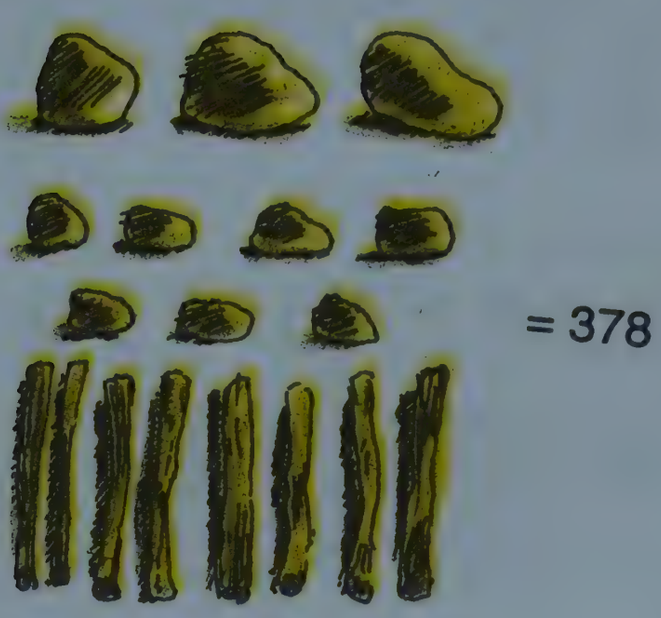

Fig. 9

Writing Numbers

The ancient civilisations in India, China, and elsewhere had developed their own systems of denoting numbers. The Indians and Egyptians used the base 10 to count numbers, e.g. they used numbers 1 to 10 and then used groups of 10 to count larger numbers. The Mayans in South America used the base 20, while the ancient Syrians used base 2 to count numbers.

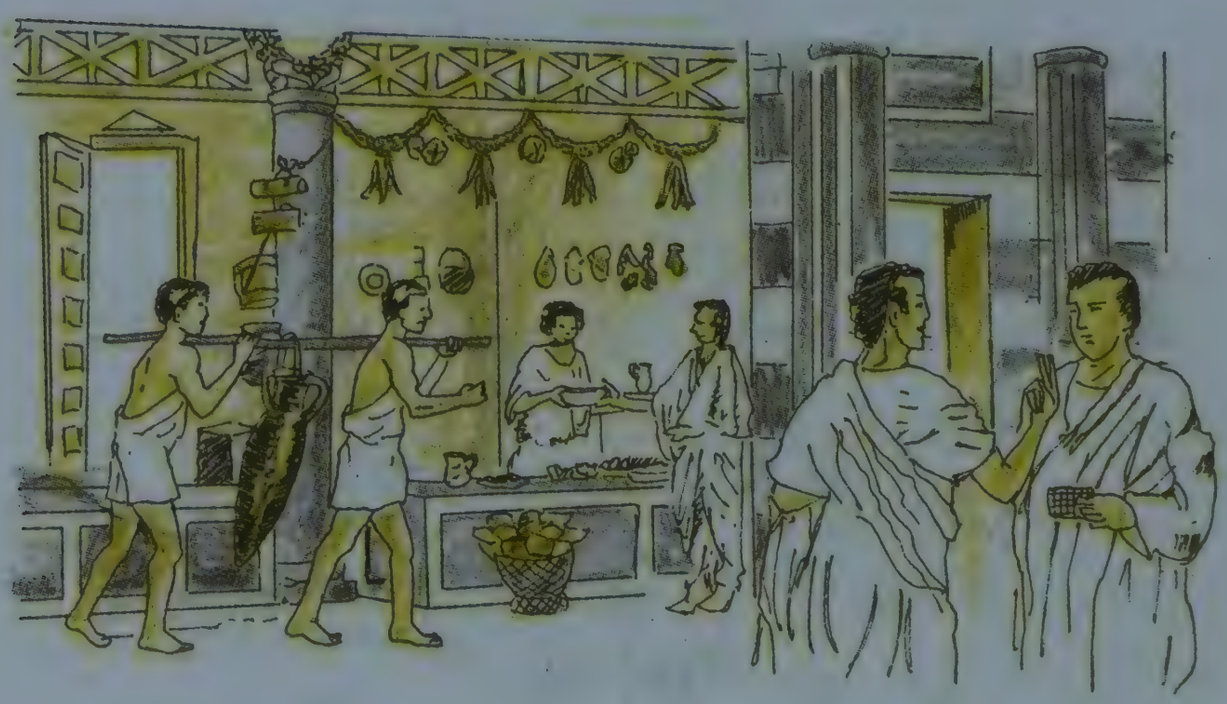

Fig. 10

The Greeks and the Romans (Fig. 10) used the letters of their alphabets, instead of separate symbols, to denote numbers. In the most widely adopted version, the Greeks

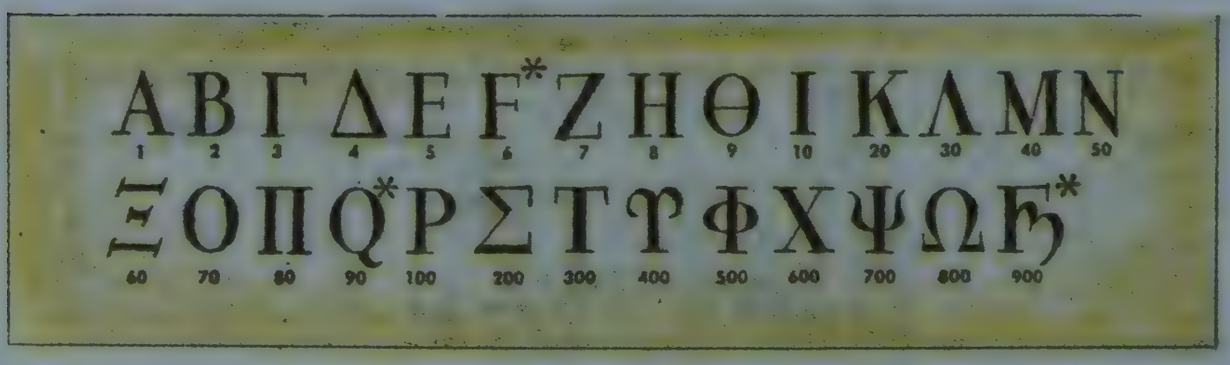

Fig. 11

used all the letters of their alphabet plus three additional symbols. Each letter stood for a definite value, e.g. the first nine symbols for numbers 1 to 9; the next nine were tens, from 10 to 90 and so on. They had no symbol for zero (Fig. 11). The Roman letters I (one), V (five), C (100), M (1,000) are well known and are still used to denote numbers (Fig. 12). However, because of this cumbersome practice, the ability of the Romans to write large numbers or do calculations was restricted.

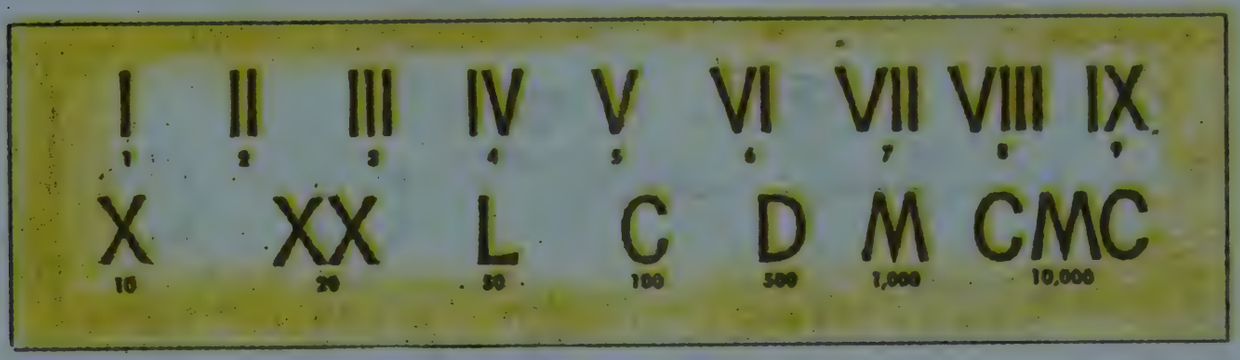

Fig. 12

It was the Hindus who developed an extremely convenient and perfect system of writing numbers, which was followed by all the world and is used even today. In the period around 100 B.C. to A.D. 200, the Hindus developed the symbols for the numbers 1 to 10 and this was later adopted by the Arabs. The Arabs conquered a large part of Spain in the eighth century and in time introduced the Hindu-Arabic numerals in the conquered land (Fig. 13). The system was gradually adopted by the other peoples of Europe. By the fifteenth century the symbols of the system had acquired the familiar form of 1, 2, 3, 4, 5, 6, 7, 8, 9, 0. In the beginning, there were separate symbols for 20, 30, 40, etc. However, in the later period two revolutionary ideas changed the face of mathematics. One was the concept of using groups of 10, 100 (i.e. 10 groups of 10), 1,000 (i.e. 10 groups of 100), etc. and giving a separate place for each group. The numerals from 1 to 9 were used to denote how many groups of 10 or 100 or 1,000 were in their respective

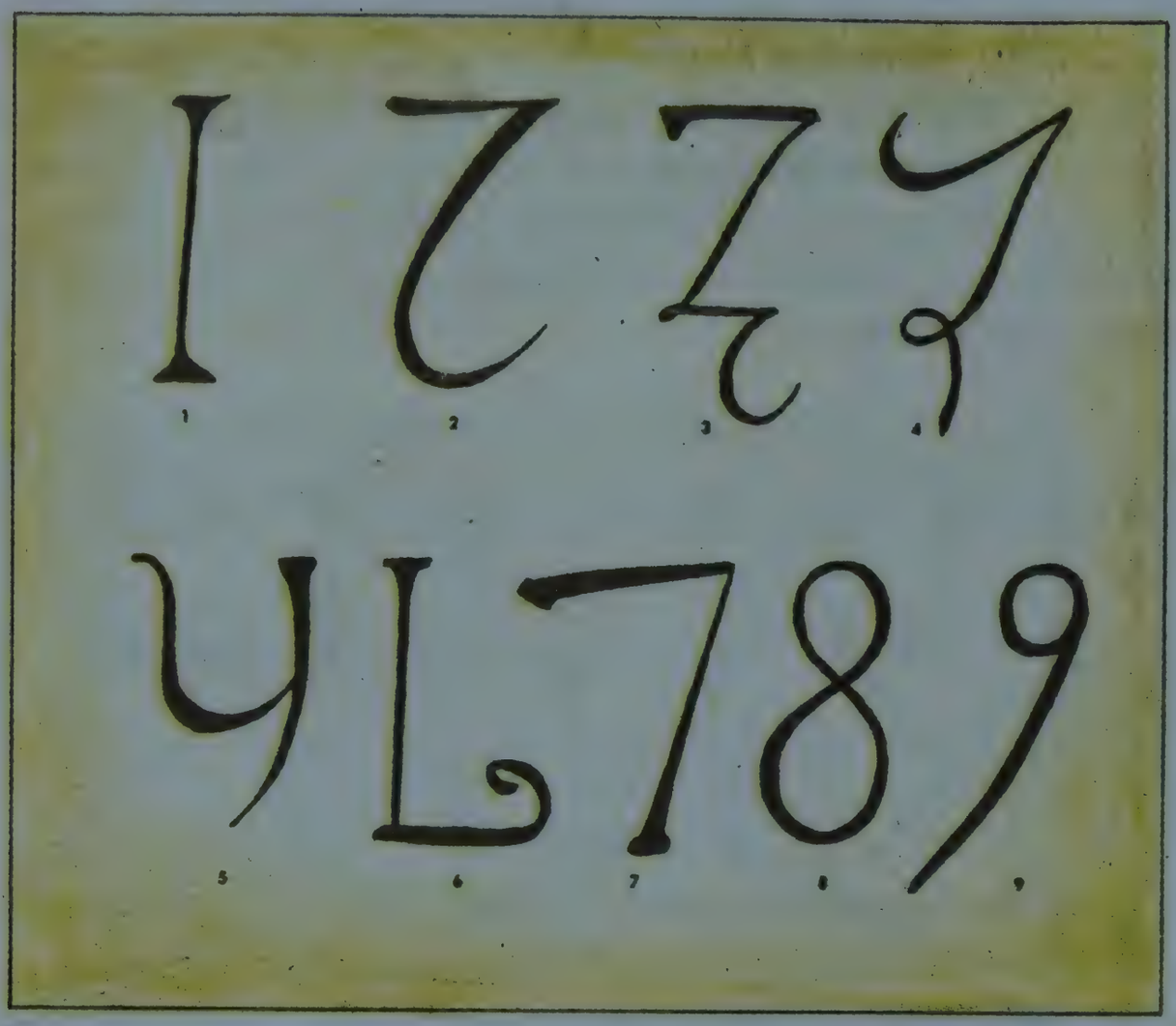

Fig. 13

places. This did away with the necessity of separate symbols for 20, 30 and so on. The concept is called the place value notation. The word for zero was *sunya*, meaning empty. The single numbers from 1 to 9 were called *ekāk*, a group of 10 was called *dashak*, a group of 100 was called *shatak* and so on. If a number contained two groups of 10 and no units then 0 was written in the units place and the number was written as 20. Similarly 307 meant three groups of 100, no group of 10 (an empty place) and seven units. The place of units group was at the right, that of 10, one place to the left of it, that of 100, one place to the left of 10's place and so on.

Thus the places of the 'digits' of a number were as shown below:

Thousands	*Hundreds*	*Tens*	*Units*
1,000	100	10	1

A digit, i.e. a numeral from 1 to 9 indicated the number of tens, hundreds, thousands and so on; zero (0) indicated an empty place. Thus, the number 85,406 would be written as shown below:

10,000	*1,000*	*100*	*10*	*1*
8	5	4	0	6

Thus 85,406 means $(8 \times 10{,}000) + (5 \times 1{,}000) + (4 \times 100) + (6 \times 1)$. That is, here the digit 8 has the place value 80,000, the value of digit 5 is 5,000 and so on.

This system of the Hindus was so revolutionary that all other systems were abandoned and it was adopted all over the world. It removed all the restrictions on writing numbers and one could write any number as large as one wished. Calculations could now be done with paper and pen instead of using mechanical devices like the abacus. In the modern index-notation the groups of 10, 100, etc. are shown as powers of 10. Hence, it is said to be based on 10, and is called the decimal system. The place values are written as

$$10^7 \quad 10^6 \quad 10^5 \quad 10^4 \quad 10^3 \quad 10^2 \quad 10^1 \quad 10^0$$

In this way the number can be extended to any number of places. In fact, the Hindus had developed it at least to 18 places. The number 1,000,000,000,000,000,00 was called a *parardha*. In the modern notation it may be called hundred thousand million millions.

Bases other than Ten

The power of the decimal system lies in its most general character. The place value concept and the concept of zero can be used to write any number in any other system with

a base other than 10. Consider a system with base 5. The places are now written as power of 5:

$$5^6 \quad 5^5 \quad 5^4 \quad 5^3 \quad 5^2 \quad 5^1 \quad 5^0$$

This system would have the digits 1, 2, 3, 4, 0. The numbers 1 to 4 are written in units place, the groups of 5 written in the next place, five groups of 5 would be written in the place of 5^2, and so on. A number 4,203 would mean four groups of 5^3, two groups of 5^2, no group of 5 and 3 units, i.e.

5^3	5^2	5^1	5^0
4	2	0	3

In the usual decimal notation this would be $(4 \times 125) + (2 \times 25) + (0 \times 5) + 3 = 553$ because $0 \times 5 = 0$. If we choose 2 as the base of our system then it consists of the digits 1 and 0. The places are represented by powers of 2; the number 553 would be written as

2^9	2^8	2^7	2^6	2^5	2^4	2^3	2^2	2^1	2^0
1	0	0	0	1	0	1	0	0	1

In this system, that is, $(1 \times 512) + (0 \times 256) + (0 \times 128) + (0 \times 64) + (1 \times 32) + (0 \times 16) + (1 \times 8) + (0 \times 4) + (0 \times 2) + 1$, shows how general the system is. More about the base 2 later.

The invention of zero is considered as the most important event in the history of mathematics. It is humorously said that the *Hindus contributed 'nothing' to mathematics.* An American writer on the history of mathematics says, "In the whole history of mathematics there has been no more a revolutionary step than the one which the Hindus took when they invented the sign 0." Laplace, one of the greatest mathematicians in the world, says, "It is India that gave us the ingenious method of expressing all numbers by means of ten symbols, each symbol receiving a value of position as well as an absolute value; a profound and important idea which appears so simple to us now that we ignore its true merit. We shall appreciate the grandeur of

this achievement when we remember that it escaped the genius of Archimedes and Apollonius, two of the greatest men produced by antiquity."

Operations

Mathematics is not only useful in everyday life; it has originated from it. Mathematics is the creation of man, who developed it in the first place to help regulate his social and personal needs. Even the life of the primitive man was full of many activities. He had to count the number of animals in his herds, the people in his village and so on. He had to measure areas to build houses, to develop the fields, to erect altars for religious functions; he had to measure volumes of the agricultural produce, weights of many objects, and most important of all, he had to measure time. For this man had developed various systems of measurement. The obelisk, used by the early Egyptians, was also known as the shadow clock. It cast the shadow of the sun;

Fig. 14

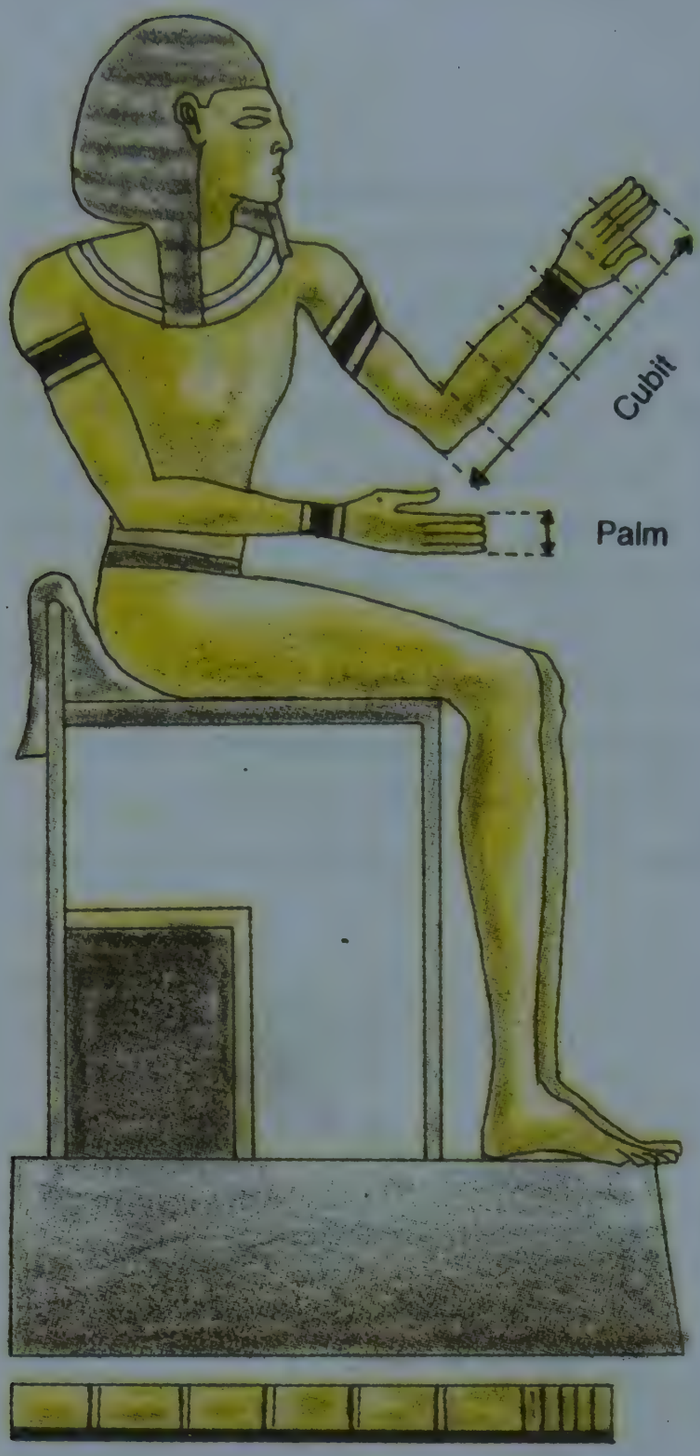

Fig. 15

the space between the shadow of the rising sun and the setting sun was divided into hours (Fig. 14). He had also developed scales to convert measurements from one system to another. The Egyptian priests used the metal bar to correspond to a cubit with the smaller divisions, palms (actually the width of four fingers) and digits indicated on the bars (Fig. 15). All these activities not only required numbers but also operations on numbers, to suit their numerous activities.

At the base of the origin of the numbers and the operations of numbers is a simple and fundamental principle. It involves one-to-one pairing of the objects in two groups. A readily available group of objects used as a reference set was the group of ten fingers on the hands. One object was paired with one finger, two objects with two fingers and so on. Thus the numbers from 1 to 10 were conceived. The size of one set of objects was compared with that of another set

of objects by pairing them off. If an object remained unpaired in a group, then that was supposed to contain more objects. If the objects were paired off perfectly, the groups were said to be equal. This sense of comparison by one-to-one pairing of objects is an innate equality which comes naturally to our mind. This idea of one-to-one pairing is now called the principle of one-to-one correspondence and is the basis for the development of mathematics. In fact, the principle of one-to-one correspondence enables us to establish a unique pair of an object and a number.

This leads to the unique identification of an object. For example, a post office and its pin code number establish a code of identification. Now it is enough to mention the pin code instead of the name of the post office to direct a letter to the proper place. Examination roll numbers determine a code between the candidate and her/his number. While declaring the results, only a list of numbers is published, instead of the names of candidates. It is because of this principle that a telephone number or the number of a ration card are used to identify a person. The working of a modern computer depends entirely on the code established in a program between the numbers and entities they represent. In the initial stages of development of numbers actual objects like bits of sticks or pebbles were used as reference sets to pair them off with the objects to be counted. The simplest and the most fundamental operation of addition suggested itself through the act of pairing of objects in two sets.

Consider a set containing three coins and another set containing two coins (Fig. 16.a). Set B is a subset of set A as one coin in it remains unpaired. Thus 3 is greater than 2. Now, if we put one more coin in set B, shown white, the coins would be paired off perfectly. The numbers now would be equal. Thus we get the relation $2 + 1 = 3$. This is in fact the beginning of operations on numbers, i.e. addition of two numbers. Thus, addition of two numbers gives another number greater than both of them. It was noticed that the addition of $2 + 1$ always gave the number 3, whatever

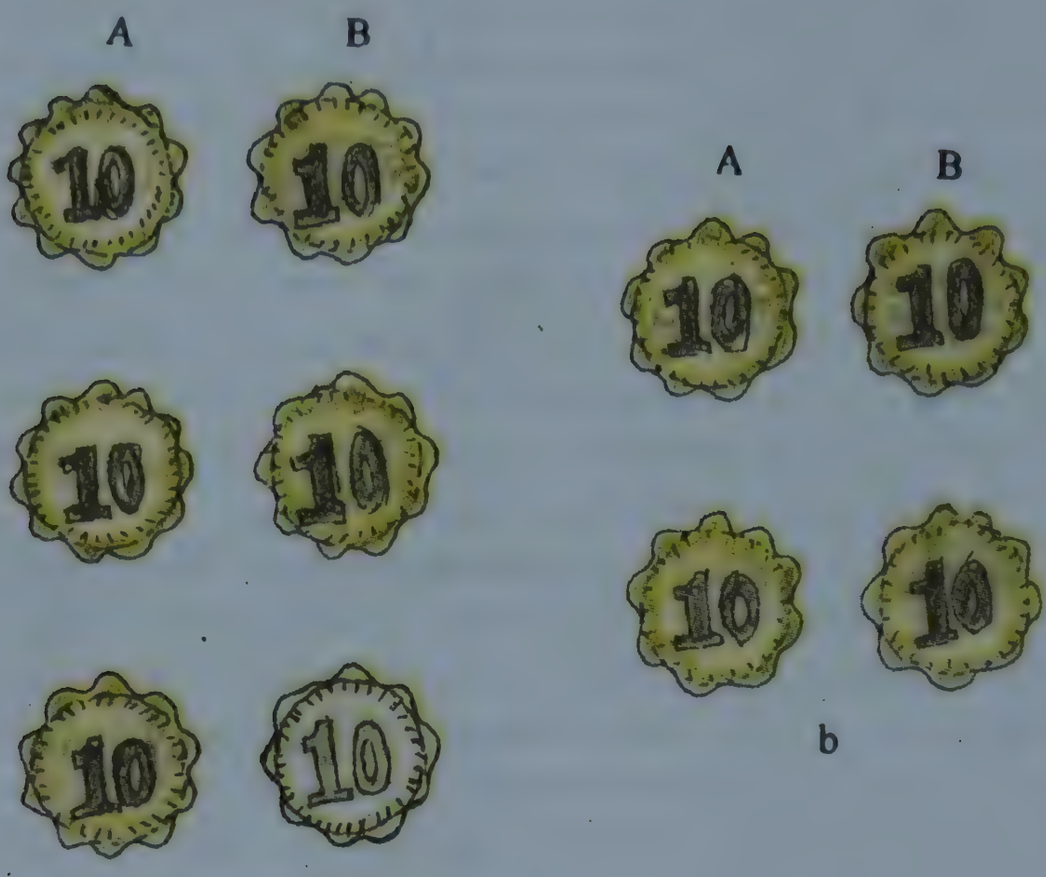

Fig. 16

be the objects. Hence, the addition gives rise to a unique number associated with a pair of numbers. Here again a one-to-one correspondence exists between 2 + 1 and 3. The set of numbers, thus extended by means of the operation of addition, consists of the numbers 1, 2, 3, 4. . ., the three dots denoting the unending sequence of these numbers. By adding any two numbers of this set, another number in the set is obtained.

Now consider the sets of coins *A* and *B* again. If we take away one coin from the set *A*, the coins can be perfectly paired as in Fig. 16. b. The act of taking away something from a set of objects suggests the operation of subtraction, 3 – 1 = 2. This is exactly the reverse of addition. The operations of addition and subtraction were the first operations carried out by man. Gradually more complex operations, like multiplication and division were devised. However, they are based on the fundamental operations of addition and subtraction. In fact, in the modern age of computers, addition and subtraction have assumed even greater importance.

Thinking in terms of groups comes naturally to our mind. Many a time we count a large number of objects in groups of two or four or five. Addition of two groups of 3 gives 6, three groups of 3 gives 9 and so on (Fig. 17).

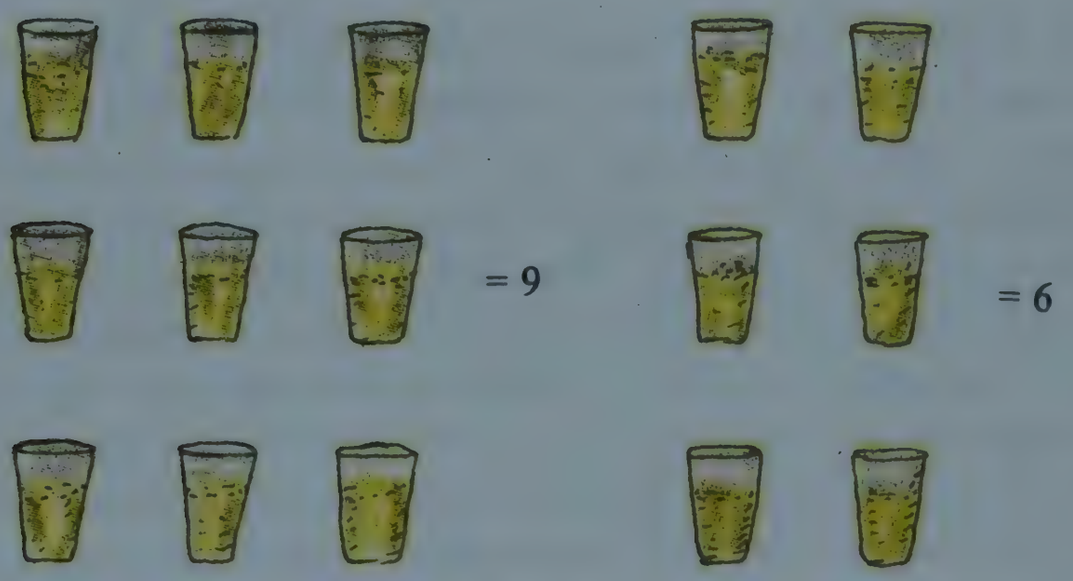

Fig. 17

This realisation leads one to the operation of multiplication. On the contrary, subtraction in groups leads to division. It would be interesting to see how addition was used to obtain the product of two numbers. The ancient Egyptians used the following method to multiply 12 by 12, by addition:

1	12
2	24
4	48
8	96

Each number in the above Table was obtained by doubling the preceding one. Now since $4 \times 12 = 48$ and $8 \times 12 = 96$, $48 + 96$ gave $12 \times 12 = 144$. It is interesting to note that they had realised the fact that $4 \times 12 + 8 \times 12 = (4 + 8) \times 12$. This fact is now known as the distributive property of multiplication.

Division consists of repeated subtraction. Consider the division $12 \div 3$. It can be accomplished by taking

away 3 from 12, four times. Thus, $12 \div 3 = 4$. Now consider the division of 19 by 8. We can take away 8 from 19 only twice. This would leave 3 to be divided by 8. Hence, $\frac{19}{8}$ can be written as

$$\frac{19}{8} = 2 + \frac{3}{8}, \quad i.e.\ 2 + \frac{2}{8} + \frac{1}{8}, i.e. \quad \frac{19}{8} = 2 + \frac{1}{4} + \frac{1}{8}$$

Division can be looked upon as dividing one whole into its parts. For example, a square piece of paper can be cut into four parts of equal size. This can be considered as dividing 1 by 4. Each part is denoted by ¼. This leads to the concept of fractions. Thus, now we can deal with two types of numbers, the natural numbers and fractions. When zero was invented, it was added to the existing numbers. The natural numbers, with zero and the fractions along with the four operations of addition, subtraction, multiplication and division are enough to carry out most of the everyday activities of our life.

It can be easily seen that when, say, 3 is added to 4, we get 7. Now to get the original number 4, we have to subtract 3 from 7. Thus, subtraction nullifies the effect of addition. Hence, subtraction is called the inverse operation of addition. Similarly, addition is called inverse of subtraction. Likewise, multiplication and division are inverse operations of each other.

Just as multiplication is repeated addition of the same number, repeated multiplication of the same number gives a square, a cube or a higher power of that number. Squaring a number is an essential operation while dealing with areas. Similarly, it is necessary to find the cube of a number while finding the volume of a rectangular solid with congruent sides. The inverse operations of taking square root and cube root are also important in everyday life.

How is it that only a few operations suffice to work out all the problems of various fields of human activity? Let us find it out with the help of an example. Consider the following problems:

- A bookseller bought 100 copies of a book at Rs 4 each. What was the cost of the books?
- What would be the total area of a house containing four rooms of 100 sq metre area each?
- How many centimetres are there in a length of 4 metres?
- What would be the total fees collected from 100 students at the rate of Rs 4 each?

It will be seen that each time the answer is 100×4. This shows that all the situations had a common pattern which is represented by the operation of multiplication, 100×4. Thus the operations are representations, or in the modern language, models of various situations that occur in everyday life. The numbers along with the operations bring out the patterns underlying our activities. The numbers in themselves are independent of the actual objects involved. Only when associated with the real physical entities do numbers get a meaning and can provide information about the entities. For example, the number 50 gives us different ideas when we say that Ramesh scored 50 marks out of 100 in mathematics, and when we hear that the maximum temperature in Jaipur was 50°C.

Development of Number System

By A.D. 500, India had become the main centre for the development of arithmetic, algebra and trigonometry (Fig. 18). The number system had been developed and the rules for various operations on whole numbers and fractions were established. The numerals from 1 to 9 and 0, the signs for operations and various terms of the language of mathematics were well defined. However, these rules were given in the form of rules for calculations and were not established as the general laws, though it seems that the Hindu mathematicians were aware of them. During this period they were also dealing with two more types of numbers, viz. negative numbers and irrational numbers. The negative numbers

Fig. 18

were introduced to deal with the problems of debt. A debt was denoted by a negative number. However, negative numbers were not yet used as freely as they are now. Negative numbers are now used to denote temperatures less than that of ice, which is taken arbitrarily as 0°C. Deficit in rainfall is denoted by negative numbers, if it is less than the average rainfall. Negative points are given to a golf player if he finishes the course in a less number of shots than the standard.

Historically, irrational numbers like $\sqrt{2}$ or $\sqrt{3}$ were known to the ancient mathematicians well before the negative numbers. They occurred while calculating the sides of a right-angled triangle and while drawing a square of a known area on the diagonal of another square, or while drawing a square having area equal to that of a given circle. For example, to draw a square having double the area of a given square, the diagonal of the given square is taken as the side of the required square (Fig. 19). Since the length of the diagonal is $\sqrt{2}$ times the side of the square, the area of

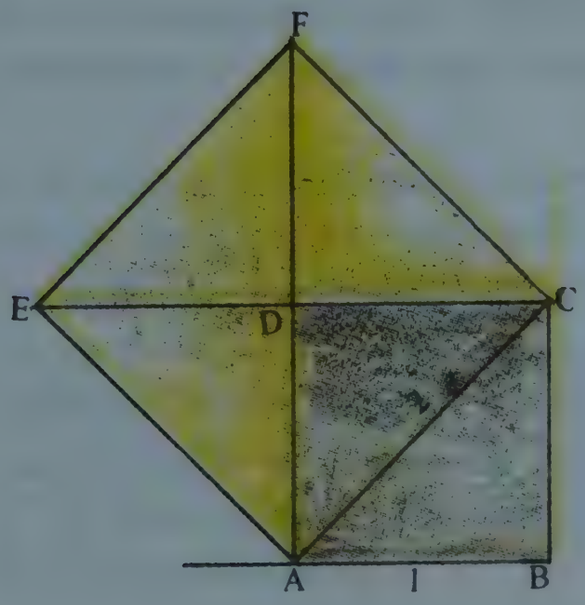

Fig. 19

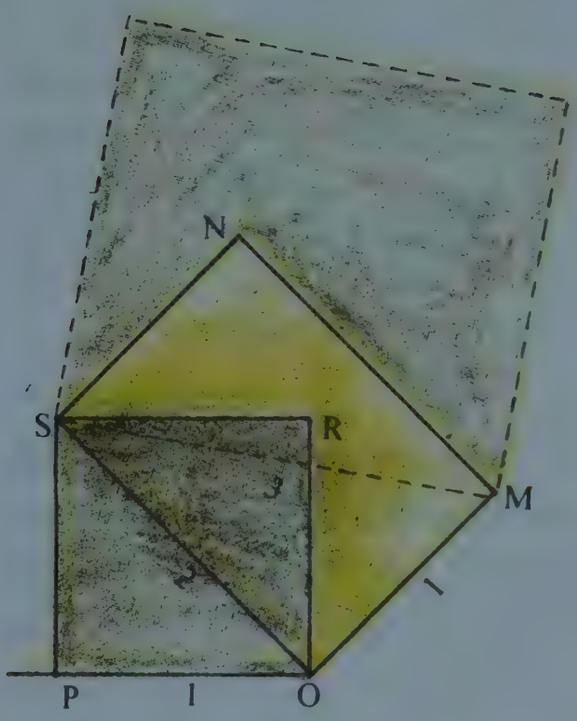

Fig. 20

the square on the diagonal is two times the area of the given square.

Similarly, to draw a square having thrice the area of the given square, a rectangle is constructed with the diagonal of the square as its length and the side of the square as its breadth (Fig. 20). Then the length of the diagonal of the rectangle is $\sqrt{3}$ times the side of the given square. The area of a square drawn on the diagonal of the rectangle is three times the area of the given square. The so-called theorem of Pythagoras was known to ancient mathematicians in India, Egypt, China and elsewhere even before the start of the Greek geometry. It was known that the exact value of $\sqrt{2}$ cannot be found. The famous mathematician, Bhaskaracharya II (A.D. 1150) had given in his famous book *Lilavati*, a method to calculate the approximate values of an irrational number like $\sqrt{3}$.

It took about 5,000 years for the complete development of the number system as we use it today. In the modern system the numbers are classified into different sets. The first set of numbers is, of course, the set of counting numbers, which are now known as *natural* numbers and

the set is designated by the letter N. It is denoted as N = {1, 2, 3, 4. . .}. The three dots indicate that there is no end to the set of natural numbers.

The next step into the development is the inclusion of zero in the natural numbers. This new set is called the set of *whole* numbers. The negative numbers, which were rejected as unnatural at one time, have now their legitimate place in the number system. The set which includes all natural numbers, zero and all-negative whole numbers is called the set of *integers* and is denoted by I. I = {. . . –4, –3, –2, –1, 0, 1, 2, 3, 4 . . .}. The dots indicate that there are infinite positive as well as negative integers.

The set of integers is further extended by the inclusion of positive and negative fractions, such as $\frac{4}{5}$ and $\frac{-8}{11}$. Any number in this set can be expressed as $\frac{a}{b}$, a ratio of two integers; for example, $\frac{2}{3}$, $\frac{-3}{5}$ and so on. Integers can also be expressed in this form. For example, $3 = \frac{6}{2}$, $7 = \frac{21}{3}$ and so on. This set is called the set of *rational* numbers, and is denoted by Q. Thus, Q = {. . . –2, $\frac{-3}{2}$, –1, $\frac{-1}{2}$, 0, $\frac{1}{2}$, 1, $\frac{3}{2}$, 2 . . .}.

New Ways of Writing Numbers

We must now study two recent systems of writing certain numbers. The first of them is the way fractions are written. The new way of writing fractions is known as the *decimal notation*. The decimal notation is seen to be simple extension of the place value notation to fractions. Observe the following place values:

. . . 10,000 1,000 100 10 1 $\frac{1}{10}$ $\frac{1}{100}$ $\frac{1}{1000}$ $\frac{1}{10000}$

Here the place values to the right of 1 are $\frac{1}{10}$, $\frac{1}{100}$ and so on. ($\frac{1}{10}$, $\frac{1}{100}$, etc. can also be written as 10^{-1}, 10^{-2}, etc.). Just as the digits in a whole number are multiplied by the value of the place and then added, the digits to the right of the unit's place in a decimal fraction are multiplied by the values of their respective places and added to get the fractional part. A dot, called the *decimal*

point, to the right of the unit's digit indicates the fractional part of the number. Thus the decimal fraction 234.567 can be written as

10^2	10^1	10^0	–	10^{-1}	. 10^{-2}	10^{-3}
2	3	4	–	5	6	7

i.e. $2 \times 100 + 3 \times 10 + 4 \times 1 + 5 \times \frac{1}{10} + 6 \times \frac{1}{100} + 7 \times \frac{1}{1000}$

In the usual notation of fractions this is written as $234\ \frac{567}{1000}$.

Writing numbers in decimal notation is extremely convenient. To add or subtract one has to write the numbers with the respective places one below the other and add or subtract like whole numbers. Multiplication is done as in the case of two whole numbers and the decimal point put in the proper place, given by the addition of the decimal places in the two numbers. Division is also done as in the case of two whole numbers and the decimal point is put when the zeros are added after the unit's digit is taken in the division.

A fraction can be converted to a terminating or a recurring decimal. Fractions with powers of ¼ = 0.25, ½ = 0.5. Fractions like $\frac{2}{3}$, $\frac{2}{9}$ are recurring, for example $\frac{2}{3} = 0.666\ldots$, $\frac{2}{9} = 0.222\ldots$ Numbers like $\sqrt{3}$, $\sqrt{2}$ can also be written as decimal fractions. However, such decimals are non-terminating and non-recurring. One gets a more and more accurate value of such a number by taking the decimal fraction to more and more places.

The second system of writing numbers is the one scientists all over the world have adopted as a standard practice for writing the values of various quantities, like length, time, force, etc. Here the number is written with only those places in the decimal fraction which are significant for the value of the quantity measured. For example, if the length is measured to the tenth of a millimetre, then the length of a segment is written as, say 27.15 cm. This is then written as 2.715×10. Thus in the decimal form of the number, integral part is only the unit's place; the rest is a fraction and

the whole number is expressed as multiplied with the proper power of 10. For example, the radius of the earth is 6,700 km. It would be written as 6.7×10^6 metres in SI units. Similarly, the radius of a bromine atom is 1.14Å. Since $1Å = 10^{-10}$ metres, it would be 1.14×10^{-10}metres. This method of writing numbers is very convenient for the comparison of two quantities. In the above example one can easily see that the radius of the earth is 10^{16} times greater than the radius of an atom. The number is so large that the ratio of 6.7 to 1.14 does not matter at all and can be safely left out of the calculation. Such a comparison is called the *order of magnitude* and is extremely important in science.

Some Strange Numbers

There are some numbers in mathematics which form the basis of modern mathematics but no one knows what they exactly are! The most well known of these numbers is the Greek letter π (pi). Popularly it is taken to be equal to the ratio $\frac{22}{7}$. It is not, however, the real value of π. In fact no one knows its exact value. All that is known about its value is that it lies between 3 and 4, which is a rather rough estimate. Still better estimate is that its value lies between 3.1 and 3.2. The estimate becomes more and more accurate as we go on taking more and more decimal places in its value. The modern value has been computed up to more than a million decimal places, with the help of computers.

The number π denotes the ratio of the circumference of a circle to its diameter. Throughout history people have used various approximations for π. For the ancient Hebrews, who built Solomon's temple, 3 was obviously good enough, for in the *Bible, First Book of Kings,* in a description of a sort of an ornamental pond outside the temple, it is said, "Also he made a molten sea of 10 cubits from brim to brim . . . and a line of 30 cubits did encompass it roundabout." The Babylonians took π to be $3\frac{1}{8}$ and the Egyptians $3\frac{12}{81}$. The ancient Hindus used the value 3.1416 to draw a square equal in area to a circle. The Chinese had taken the value of

π equal to 3.16, although Tsú Ch'ung Chih had calculated it correct to seven places to be between 3.1415926 and 3.1415927. Although the number π was used as a ratio in the ancient times, it was not considered as a special number and the values obtained were obtained indirectly. The special status of π was first given by Archimedes and its value was calculated as the ratio of circumference to the diameter of a circle. He decided that its value was between $3\frac{10}{71}$ and $3\frac{1}{7}$. Hence, it was taken to be approximately equal to $3\frac{1}{7}$, i.e. $\frac{22}{7}$. He did not, however, denote it by any letter. The letter was first used to denote this ratio by William Jones in 1706. An interesting one was hit upon by accident in 1964 (the year British Television changed to 625 lines on the screen) by an engineer working for Pye Television: $\frac{1964}{625}$. However, just out of interest, since the year 1949, mathematicians have programmed computers to calculate π to more and more decimal places, and in 1984 a computer in Japan worked out two million decimal places. Obviously numbers are important for many practical reasons, but they also have a fascination of their own, and the branch of mathematics, known as *number theory* or *higher arithmetic*, is concerned entirely with numbers and how they behave.

The number π is the basis of circular (or trigonometric) functions. Angles are measured in a unit called *radian*. 2π radians equal 360°. Hence, all trigonometric functions are expressed in terms of π. It is required to find out the circumference of a circle and hence is the basis of modern technology. Electrical engineering, propagation of waves of all kinds, astronomical calculations, are all based on trigonometric functions. Thus π is an important number in modern sciences.

Another strange number similar to π is e. Like π the number e also forms the basis of modern mathematics. Functions like trigonometric functions, hyperbolic functions, expotential functions are based on e.

The number e is obtained by considering the sum of the infinite series:

$$1 + 1 + \frac{1}{1.2} + \frac{1}{1.2.3} + \frac{1}{1.2.3.4} + \ldots$$

This is an unending series. However, the sum is always less than 3. Both π and e are irrational numbers.

Yet another strange number is i. It is called an *imaginary number* because it cannot really exist, as it is taken to be the square root of a negative number such as $\sqrt{-1}$. All numbers have one common property: square of any number, positive or negative, is always positive. Thus, in the real number system it is impossible to find a number, square of which would be negative. However, mathematicians since ancient times had encountered such numbers while finding the square roots of numbers. The name 'imaginary' number was given to such numbers by Des Cartes. Any number of the form $a + b\sqrt{-1}$, i.e. $a + bi$ is called a *complex* number. Complex numbers are used extensively in modern mathematics and form the basis of many functions used in mathematics and science.

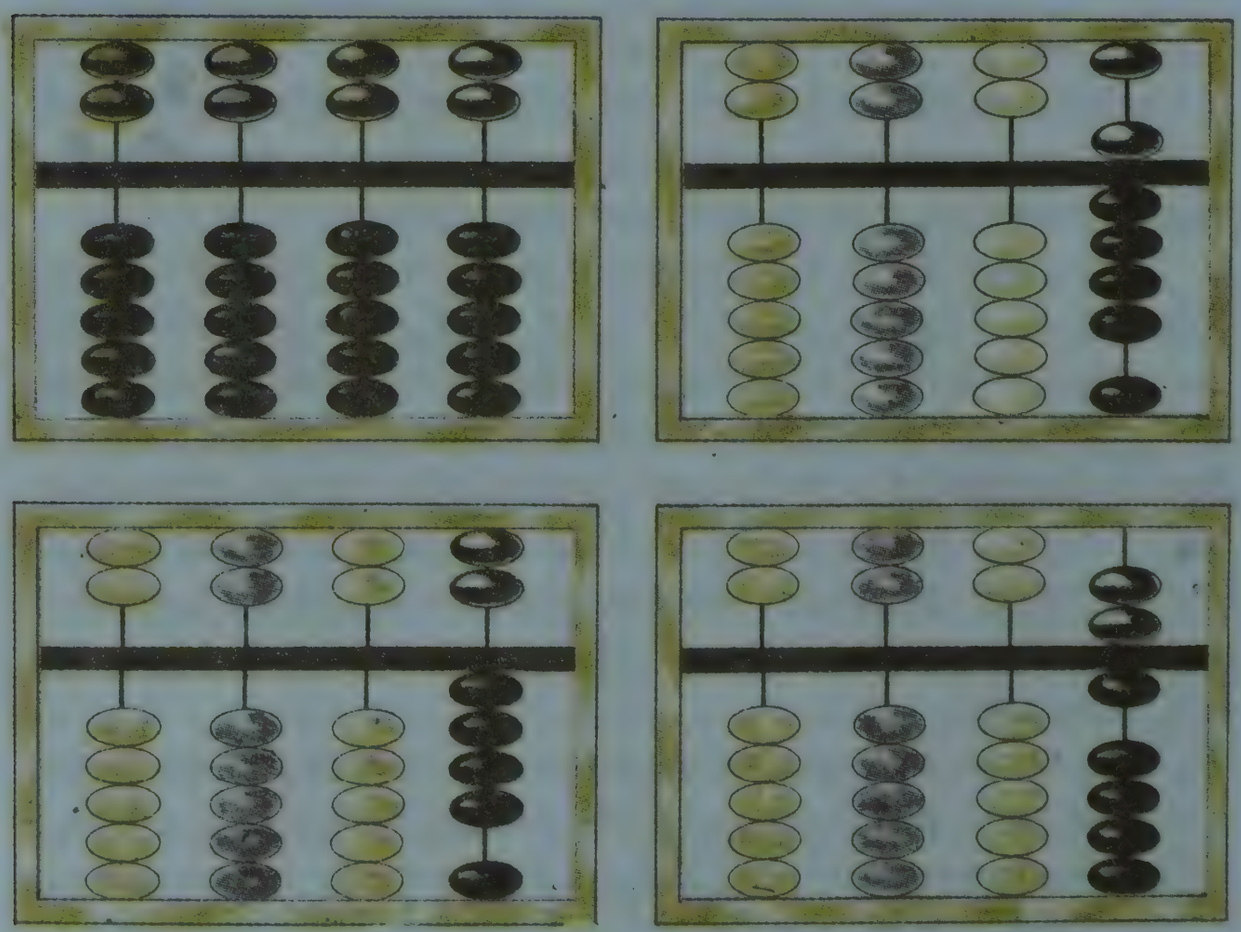

Fig. 21

Calculating Numbers

The progress of civilisation has gone hand in hand with the progress in counting numbers. For example, progress in navigation followed the invention of the logarithmic and trigonometric tables, and technology progressed when the slide rule was ready, while the modern age is known as the Age of Computers.

The abacus (Fig. 21) was the first device of a permanent nature which was used for calculating. It consisted of vertical bars fitted in a frame. A horizontal bar separated the frame into two parts. The upper part contained two beads while the lower part contained five beads on each bar. The upper beads denoted fives and the lower one units. This was used to add and subtract large numbers.

Although the origin of abacus is not known, it was used in China for the last 3,000 years and is still popular in China and Japan.

Multiplication

Multiplication is nothing but adding the same number again

	1	2	3	4	5	6	7	8	9	10
1	1	2	3	4	5	6	7	8	9	10
2	2	4	6	8	10	12	14	16	18	20
3	3	6	9	12	15	18	21	24	27	30
4	4	8	12	16	20	24	28	32	36	40
5	5	10	15	20	25	30	35	40	45	50
6	6	12	18	24	30	36	42	48	54	60
7	7	14	21	28	35	42	49	56	63	70
8	8	16	24	32	40	48	56	64	72	80
9	9	18	27	36	45	54	63	72	81	90
10	10	20	30	40	50	60	70	80	90	100

Fig. 22

and again. Thus for 2 we get the series of multiples of 2, such as

2, 4, 6, 8, 10, 12, 14, 16, 18, 20

up to 2×10. We can go on writing the higher multiples. However, it is not necessary to go beyond 2×10. For if we say 2×16, we can get it by using the Table up to 10; thus $2 \times 16 = 16 \times 2 = (10 + 6) \times 2 = 20 + 12 = 32$. The Table given in Fig. 22 is sufficient to carry out all multiplications and divisions that are usually required.

In India multiplication Tables were developed to a large extent. Tables for integers up to 30 for first 10 multiples, and for fractions for first 100 multiples were developed and used extensively for more than 2,000 years.

Tables for multiplication are also used for division. For example if $3 \times 4 = 12$, then $12 \div 4 = 3$. Multiplication Table is a powerful device for speedy and accurate calculations except for very complicated calculations which involve powers or roots of numbers.

Logarithms

During the fifteenth century sea voyages were undertaken by the western countries like Great Britain, Denmark, Spain and Portugal on a large scale. In sea, navigation depended on knowing the correct positions of stars, planets and constellations. It was necessary to prepare accurate trigonometric Tables, for which complicated calculations were necessary. Logarithms were invented to facilitate the calculations.

The logarithms convert the problem of multiplication into a problem of addition and a problem of division into a problem of subtraction. This immensely simplifies the task of calculation.

Computers

The electronic computer is the most efficient of all the machines developed till date for computation. It can handle large numbers, is extremely fast and accurate. At the heart

of a computer are electronic circuits which carry out the logical operations required for computations. However, a computer does not work with the language of words. It works with the language of numbers and all the words in computer language consist of only two numbers—1 and 0. Since the binary system consists of only two digits, 1 and 0, all numbers are expressed in terms of powers of 2. The place values are as shown below:

$$2^8\ 2^7\ 2^6\ 2^5\ 2^4\ 2^3\ 2^2\ 2^1\ 2^0$$

Using this place value system, numbers in the 10-base system can be converted to the 2-base system. For example,

1	=	1
2	=	10
3	=	11
4	=	100
5	=	101
6	=	110
7	=	111
8	=	1,000
9	=	1,001
10	=	1,010 and so on.

In a computer circuit 1 means a closed circuit and 0 means an open circuit. A code is fed to the computer through a procedure called a *program*. The code translates ordinary language into the number language of the computer and vice versa. While doing computations, a computer converts a problem of multiplication into a problem of addition and a problem of division into that of subtraction. This seems to be going back to a primitive procedure. However, a computer does these additions or subtractions so fast that nobody minds it. Nowadays highly sophisticated computer programs are available which can deal with all sorts of complicated problems—from solving hundreds of equations to drawing pictures and printing books, even booking reservations. Indeed, computers have become the mainstay of modern life. It may be said that we are living in the world of one and none!

3

THE VARIABLES

Mathematics deals with situations in everyday life. The situations are, however, varied in nature.

We come across many instances of unknown or variable quantities in the course of a day. For example, when we travel in a bus, the speed of the bus keeps on changing; the number of passengers changes at every stop; and the amount in the purse of the bus conductor also keeps changing. In fact, life is full of variable quantities. The monthly expenditure of a person, the profit earned in a business, the yield of a crop, the yearly rainfall, the temperature of air, the height of a plant and even the length of a day are all variable quantities.

The Hindu mathematicians were the first to recognise the need for denoting unknown or the variables and used a different notation for them. The world *yavat-tavat*, meaning 'as much' or 'as many', was used to denote an unknown quantity in a manuscript dating back to the period well before 300 B.C. Later mathematicians used the word *avyakt*, meaning 'an unknown quantity' (as against *vyakt* which denotes a known quantity) instead of *yavat- tavat*.

Recent discoveries have shown that Babylonians solved problems in variables though they had no symbols for them. They used only words to denote such numbers, and that is why their algebra has been referred to as rhetorical algebra. The *papyrus*, an Egyptian scroll going back to 1600 B.C., has a number of problems in algebra, in which the

unknown is referred to as a *hau*, meaning 'a heap'.

Aryabhatta I, Brahmagupta and Bhaskaracharya II were some of the eminent ancient Hindu mathematicians who propounded and perfected the methods of *beej-ganita*, the mathematics of variables, as it is expressed in modern language. *Beej-ganita* means literally, 'the mathematics of letters'.

Little further progress was made in this field until we came to Diophantus, a Greek mathematician, living in third century A.D. He reduced problems to equations, representing the unknown quality by a symbol suggesting the Greek Σ (sigma). He also introduced an interesting system of abbreviations in which he used only the initial letters of words and omitted all unnecessary words. In the sixteenth century, Francois Vieta, a French mathematician, used the vowels a, e, i, o, u to represent unknown numbers and the consonants b, c, d, f, g and so on to stand for values that remained fixed throughout a given problem. Rene Descartes, the great sixteenth-century French philosopher, proposed the system of a, b, c and other letters near the beginning of the alphabet to represent the fixed values. The concept of using letters to denote variables or unknown quantities in a problem was the first step towards generalisation of the basic concepts of arithmetic and the key to the progress of mathematics.

The word 'algebra' originated from the title of a work on algebra by a Persian, Mohammed ibn Mesa al-Kwarizmi who lived in ninth century A.D. He wrote in Arabic a work called *Al-jebr w'al Muqabala*, which means 'restoration and reduction'. By *al-jebr*, or restoration, was meant transposing of negative terms to the other side of an equation to make them positive. When the Arabs came to Spain, they brought this word with them. In the course of time *al-jebr* got changed to 'algebra', and the word came to be applied not to a single operation, but to the many operations involved in algebra.

Mathematical Patterns

The most important function of algebra is to bring out a pattern in its general form. For example, consider the arithmetical operation of addition. Take the following example:

$$\begin{array}{lcll} 3+4 & = & 7 & = & 4+3 \\ 5+10 & = & 15 & = & 10+5 \\ 6+7 & = & 13 & = & 7+6 \text{ and so on.} \end{array}$$

Deleting the numbers in the middle, we get

$$\begin{array}{lcl} 3+4 & = & 4+3 \\ 5+10 & = & 10+5 \\ 6+7 & = & 7+6 \end{array}$$

If we replace the numbers in these expressions by letters, we get $a + b = b + a$. This single algebraic expression represents all the possible expressions of the above kind. The algebraic expression brings out the particular property of the operation of addition—that the result of addition is the same, if the order of addition of two numbers is reversed. This property of addition is known as the *commutative law of addition*. The following laws express the most important properties of addition and multiplication that are constantly used in everyday calculations. In these expressions a, b, c are any numbers.

$a + b = b + a$	Commutative law of addition
$(a + b) + c = a + (b + c)$	Associative law of addition
$a \times b = b \times a$	Commutative law of multiplication
$(a \times b) \times c = a \times (b \times c)$	Associative law of multiplication
$a \times (b + c) = a \times b + a \times c$	Distributive law of multiplication with respect to addition

As has been mentioned earlier, the discovery of zero and its properties was the most important event in the history of mathematics. Aryabhatta and other Hindu mathematicians had studied the peculiar properties of zero. They had, of course, exposed the properties verbally in verse form, as was the prevalent practice in India at that time.

In the modern notation the properties are:

$$a + 0 = 0 + a = a$$
$$a \times 0 = 0 \times a = 0$$
$$a - 0 = a$$
$$a \div 0 \text{ is indeterminate.}$$

However, they were not very clear about the division by zero.

Let us see how these laws of operations help us in our everyday chores. One of the most important activities is to keep the account of various transactions of money. Suppose, for example, the amounts of Rs 50, Rs 100, Rs 45 and Rs 40 are to be added, i.e. the sum of 50 + 100 + 45 + 40 is required. The numbers are always added two at a time. The addition goes as follows:

$$\begin{aligned} 50 + 100 + 45 + 40 &= (50 + 100) + 45 + 40 \\ &= (150 + 45) + 40 \\ &= 195 + 40 \\ &= 235 \end{aligned}$$

However, it would not matter at all if the addition is done as follows:

$$\begin{aligned} 50 + 100 + 45 + 40 &= 50 + 100 + (45 + 40) \\ &= 50 + (100 + 85) \\ &= 50 + 185 \\ &= 235 \end{aligned}$$

Thus the numbers can be added in any order. Here both the commutative and associative laws are freely used mostly intuitively, in our daily life. The same is the case with multiplication. The multiplication 5×17 is usually done as 17×5 since it is easier to find it from the multiplication Table. The product $12 \times 5 \times 2$ is done as follows:

$$\begin{aligned} 12 \times 5 \times 2 &= (12 \times 5) \times 2 \\ &= 60 \times 2 \\ &= 120 \\ \text{or} \quad &= 12 \times (5 \times 2) = 12 \times 10 = 120 \end{aligned}$$

Sometimes this helps in carrying out a difficult multiplication; for example, it is easier to find the product $137 \times (5 \times 2)$, i.e. 137×10 than $(137 \times 5) \times 2$. The distributive law of multiplication, viz. $a\,(b + c) = ab + ac$ is used while we multiply large numbers. For example, the product 256×7 is carried out in the following way:

$$\begin{aligned}(256 \times 7) &= (200 + 50 + 6) \times 7 \\ &= 1400 + 350 + 42 \\ &= 1792\end{aligned}$$

In actual practice the product is done orally by carrying forward digits denoting the groups of 10 or 100 at each step. It can be easily seen that these laws are an integral part of our everyday life.

The Equality Relation

When the letters in an algebraic expression are replaced by numbers, a value of the expression is obtained. If the values of two expressions are found, it may happen that the two expressions have the same value. Such expressions are said to be *equivalent* (having equal values). It is said that an *equality relation* exists between the two expressions. This relation is written in the form of an equation. For example, $a \times (b + c)$ and $a \times b + a \times c$ are two equivalent expressions.

$$\begin{aligned}\text{Consider the value } a &= 2,\ b = 3,\ c = 4;\ \text{then} \\ a \times (b + c) &= 2 \times (3 + 4) \\ &= 2 \times 7 \\ &= 14 \\ \text{and } a \times b + a \times c &= 2 \times 3 + 2 \times 4 \\ &= 6 + 8 \\ &= 14\end{aligned}$$

Hence the relation is expressed as $a \times (b + c) = a \times b + a \times c$.

There are various types of equivalent expressions. They form a large part of our daily life as well as of the study of mathematics and science. Consider the following problem. A football club was joined by three new members.

After a month, half of the existing members left. How many members were there in the beginning?

Let x be the number of members initially. When three members joined the club, the number became $x + 3$. When half of them left $(x+ 3)/2$ members remained. When 10 members joined this club the number was $[(x + 3)/2] + 10$. This was equal to x. Hence we get

$$\begin{aligned} \tfrac{x+3}{2} + 10 &= x \\ 2\left[\tfrac{x+3}{2} + 10\right] &= 2x \\ \therefore\ x + 3 + 20 &= 2x \\ \therefore\ x + 23 &= 2x \\ \therefore\ x + 23 - x &= 2x - x \\ 23 &= x \end{aligned}$$

Hence, initially there were 23 members in the club.

Systems of Equations

Bhaskaracharya has given the following problem in his famous book, *Lilavati*: "Tell me, oh student, what are the numbers whose sum is 101 and the difference 25, if you know how to do it." Let us try to solve the problem, using equations. Let the two numbers be x and y. Then

$$\begin{aligned} & x + y = 101 \\ \text{and} \quad & x - y = 25 \\ & (x + y) + (x - y) = 101 + 25 \\ & 2x = 126 \\ & x = 63 \\ \text{Now} \quad & x + y = 101 \\ \text{i.e.} \quad & 63 + y = 101 \\ & y = 101 - 63 \\ & y = 38 \end{aligned}$$

Thus 63 and 38 are the numbers.

Sequences

Two friends *A* and *B* were having a chat. *A* made an offer to *B*, the other, "I will pay you one thousand rupees on the first day, 2,000 rupees on the second day, 3,000 rupees on

the third day, and so on for a month. In return you should pay me one paisa on the first day and double the amount next day and so on for every day for a month." *B* accepted the offer readily. But to his chagrin, he had to repent afterwards. Can you guess why?

Let us see what each of them would get each day.

Day	*A (rupees)*	*B (rupees)*
1	1,000	.01
2	2,000	.02
3	3,000	.04
4	4,000	.08
5	5,000	.16
6	6,000	.32
7	7,000	.64
8	8,000	1.28
9	9,000	2.56
10	10,000	5.12
11	11,000	10.24
12	12,000	20.48
13	13,000	40.96
14	14,000	81.92
15	15,000	163.84
16	16,000	327.68
17	17,000	655.36
18	18,000	1310.72
19	19,000	2621.44
20	20,000	5242.88
21	21,000	10485.76
22	22,000	20971.52
23	23,000	41943.04
24	24,000	83886.08

Do you see now why *A* had to repent? On the 23rd day he had to pay almost double the amount he received from *B*.

Can you tell how *B* could overtake *A* even if he had started with just one paisa? Let us see how the numbers went on growing:

	Day 1	*Day 2*	*Day 3*	*Day 4*
For *A*	1000	2000	3000	4000 . . .
i.e.	1000	1000+1000	2000+1000	3000+1000 . . .
For *B*	0.01	0.02	0.04	0.08 . . .
i.e.	0.01	0.01 × 2	0.02 × 2	0.04 × 2 . . .

The numbers are in a sequence and grow in a systematic way. In the first sequence each number is greater than the previous number by 1,000. In the second sequence each number is double the previous one. One can see that by addition, the sequence grows slowly, while by multiplication it grows very fast. The sequence of the first type is called *Arithmetic Progression* (A.P.); the second type is known as a *Geometric Progression* (G.P.). The study of sequences like these is an important part of algebra.

Series: Adding the Terms of a Sequence

In an elementary class a teacher gave the children a difficult task of adding all the numbers from 1 to 100. He was sure of getting a rest for some time. He was in for a surprise, though. As soon as he finished giving the sum, a boy wrote the answer on his slate and put it down. The teacher asked the boy to show his answer. The answer was 5,050, which was the correct answer. The boy was Freidrich Gauss, who later became one of the greatest mathematicians. Instead of adding all the numbers, he found out that the sum is given by (1+100) + (2+99) + (3+98) +. . . ., that there were 50 pairs each adding to 101; so the sum was 50 × 101= 5,050. This is in fact the method used to find the sum of a series in arithmetic series. The sum of an arithmetic series of n numbers with a as the first term and d as the common difference is given by

$$S_n = \frac{n}{2} [2a + (n-1)\, d]$$

If we put $a = 1$, $d = 1$ and $n = 100$, we get

$$S_{100} = \frac{100}{2} [2 + 99] = 50 \times 101 = 5{,}050$$

Gauss's genius had found out the pattern of numbers in the addition in a flash.

The sum of a geometric series with a as its first term and common ratio r, to first n terms is given by the formula

$$S_n = \frac{a(r^n - 1)}{r - 1}$$

We can now find out what each of the friends A and B received at the end of the month. The amount received by A is given by

$$\begin{aligned} S_n &= \frac{30}{2} [2 \times 1000 + (30 - 1)\, 1000] \\ &= 15\, [2000 + 29000] \\ &= 15 \times 31000 \\ &= 465000 \text{ or Rs } 4{,}65{,}000 \end{aligned}$$

The amount received by B, if given by

$$\begin{aligned} S_n &= \frac{1(2^{30} - 1)}{(2 - 1)} \\ &= 2^{30} - 1 \\ &= 1073741823 \text{ or Rs } 1{,}07{,}37{,}418.23 \end{aligned}$$

Mathematics in Real Life

Mathematics can be said to be a mirror in which life is reflected. A mathematician facing a real problem can then, like Alice in Wonderland, wander in the mathematical image of life, allow his imagination to work, solve the mirror problem and then return back to real life with the real answer. Let us take a simple problem and see how mathematics helps to solve it.

> Expenses of a student's hostel increased from Rs 7,000 to 9,000 per month, when the number of boarders increased from 12 to 16. The hostel had to pay a fixed rent per month and spend a certain amount per person as living charges. What was the rent and the expense per person?

To convert the real problem into a mathematical one, let us denote the quantities with letters. Let M be the number of students, A be the amount charged per person, R the rent and E the total expense per month. Then we get

$$E = M \times A + R$$

Now, according to given conditions, substituting the values for M and E we get

$$7000 = 12A + R \ldots\ldots (1)$$
$$\text{and} \quad 9000 = 16A + R \ldots\ldots (5)$$

This is now a mathematical problem of solving two simultaneous equations. Subtracting (1) from (2) we get

$$2000 = 4A$$
$$A = 500$$

Substituting in (1), we get

$$7000 = 6000 + R$$
$$R = 1000$$

Here the mathematical part of the problem is over. We have now to interpret the result into the real life. We then conclude that the amount spent per person is Rs 500 and the rent is Rs 1,000.

To solve the above problem we used letters to denote various quantities of the problem and formed the equation

$$E = M \times A + R$$

which is in fact a mathematical model of the problem. This method is called *mathematical modelling*, which provides us with a very powerful means of solving problems in real life. Today, with the help of computers, many versions of a mathematical model can be tested by changing the values of the variables and the most suitable model is selected to deal with extremely complicated problems in meteorology, aeronautics, hydrodynamics, nuclear physics, social sciences like economics, sociology and communication.

4

RATIOS: COMPARING THINGS

In everyday life we often compare two things while buying a thing, or choosing between two jobs, and so on. Comparisons like 'Calcutta is bigger than Bombay', 'Subhash is stronger than Satish', 'Kalpana is better than Kavita in mathematics' are always made. However, these statements are rather vague. They do not give the exact idea of how much bigger, or how much stronger. One can compare things properly if numbers are assigned to them. Then one can compare the numbers according to the established rules of mathematics and arrive at a proper conclusion. For example, if one knows that the populations of Calcutta and Bombay are 10 million and 8 million respectively, then one can say that Calcutta is bigger than Bombay since 10 million is larger than 8 million. Thus to compare two things, we compare numbers assigned to them. The problem thus reduces to comparing two numbers.

There are two ways of comparing two numbers. Suppose a and b are the two numbers. Then we can find the difference $a - b$. For example 14 - 10 = 4, i.e. 14 is greater than 10 by 4. The other way of comparing two numbers is to divide one number by the other. Suppose 20 and 10 are two numbers, then $\frac{20}{10} = \frac{2}{1}$. Thus 20 is two times 10 or $\frac{10}{20} = \frac{1}{2}$; that is 10 is $\frac{1}{2}$ of 20. Thus, in general, a ratio answers the questions, "How many times one number is of the other?" or "What fraction of one number is the other?". Another way of saying this is that the ratio of 20 to 10 is

2 to 1. Comparing two numbers by taking their ratio is quite common and also very useful. For example, it is better to say that 50 is $\frac{1}{2}$ of 100 than to say that 50 is less than 100 by 50.

Ratios are expressed in different ways. For example, a company report while extolling the efficiency of the company says: "The profits have doubled this year." This means that the ratio of this year's profit to last year's profit is $\frac{2}{1}$. While distributing the bonus shares a company says that it would distribute the shares in the ratio 1:2. That means, one bonus share will be given for every two shares held by the shareholder. Talking about the literacy of a village, it is said that there is one literate for every three illiterates in the village. That is, the ratio of literates to illiterates is $\frac{1}{3}$. The ratios are sometimes written in a different notation. For example, the ratio $\frac{1}{3}$ in the above example is also written as 1:3. A ratio $\frac{2}{3}$ can be written as $2 \div 3$ or $\frac{2}{3}$ or 2:3, and so on.

When a ratio is written in the form $\frac{a}{b}$ where a and b are integers and b is not equal to zero, it can be treated as a rational number. Hence, when we are dealing with ratios, rules of rational numbers are applied to them. For example, $\frac{2}{3}$ and $\frac{7}{8}$ are two ratios. It is necessary to decide which is the greater one, then they can be compared as if they are two fractions. Hence, writing that $\frac{2}{3} = \frac{16}{24}$, and $\frac{7}{8} = \frac{21}{24}$, it can be found that $\frac{7}{8}$ is the greater of the two.

Care has to be taken when quantities compared are not simple numbers, but are measured in some units. For example, "Which is greater, 150 minutes or 3 hours?" Here, the quantities are in different units and cannot be compared unless they are reduced to the same units. Three hours means 180 minutes. Now 150 minutes and 180 minutes can be compared. On the other hand, 150 minutes means 2.5 hours. Now 2.5 hours and 3 hours can be compared.

Percentages

In practice ratios are used and compared in various ways. A student gets 75 marks out of 100 in language and 120 out of 150 in science. In which subject is his performance better? We can compare the ratios $\frac{75}{100}$ and $\frac{120}{150}$ and decide the issue: $\frac{75}{100} = \frac{225}{300}$ and $\frac{120}{150} = \frac{240}{300}$. Hence by comparing $\frac{225}{300}$ and $\frac{240}{300}$ we can see that he had done better in science than in language. Here we have taken 300 as the common denominator in both the ratios. In principle any number can be taken as the common denominator, and the ratio is expressed as a percentage. For example, 75 marks out of hundred is expressed as 75 per cent marks. The words per cent mean 'for each hundred'. In this way $\frac{120}{150} = \frac{80}{100}$. Hence 120 out of 150 mean 80 out of 100, i.e. 80 per cent. The ratio in terms of 100 means 'percentage'. In everyday life, marks in examinations, the results of various examinations, number of literates and illiterates in a population, and many other quantities are expressed as percentages.

Proportion

We often hear expressions like 'the more the merrier', 'the more one gets the more one wants', 'the deeper you go into the water the higher the pressure gets', or 'it gets cooler as you go up in the air'. Such expressions tell us about relations between two things. If one of the things changes, the other also changes, either in the same way or in the opposite way. For example, as the depth under water *increases*, the pressure of water also *increases* and vice versa. Here the quantities 'depth' and 'pressure' increase or decrease together. On the other hand, as the height of a point from sea level increases, the temperature of air at that point decreases. Thus the changes in the height and the temperature of air are opposite to each other.

Let us take a few examples where two related things change or vary in the same manner. Consider the number of copies of a book bought and the price paid for them.

No. of Books	*Price Paid*
10	200
20	400
50	1000

Here as the number of books increases, the total price paid also increases. Thus we consider the ratios $\frac{10}{20}$ and $\frac{200}{400}$. It is seen that $\frac{10}{20} = \frac{200}{400} = \frac{1}{2}$. Similarly $\frac{10}{50} = \frac{200}{1000} = \frac{1}{5}$. Thus in each case the ratios of corresponding values of the quantities concerned are equal. If two ratios are equal, the four quantities involved in the ratios are said to be proportional. In general if $\frac{a}{b} = \frac{c}{d}$ then a, b, c, d are said to be in proportion, when written in that order. This means that though a, b, c, d are in proportion, a, d, b, c are not. On the other hand, given that a, b, c, d are in proportion, $\frac{a}{b} = \frac{c}{d}$. The numbers a, b, c, d, are called the *terms of proportion*. When three terms of a proportion are known, the remaining one can be found out. The following problem fits into this pattern and can be solved in the same way.

A person bought 3 copies of a book for Rs 75. How many copies can he get for Rs 300? If x is the number of copies, we get

$$\frac{3}{75} = \frac{x}{300}$$
$$x = \frac{3}{75} \times 300$$
$$= 12$$

Thus the person gets 12 copies of the book.

Consider another example. For a car travelling at an average speed of 50 km per hour, the distance covered would increase with time, as in the following Table:

Time (hrs)	*Distance (km)*
1	50
2	100
3	150
4	200

In the above examples the related quantities, viz.

number of books and the price, and time and distance are changing in the same manner, i.e. increasing together. Such quantities are said to be in *direct proportion*. Sometimes the related quantities change in an opposite manner, i.e. while one increases, the other decreases or vice versa. Consider the following problem:

> An amount of Rs 1,000 is to be repaid by monthly instalments. The time required to repay the loan depends upon the amount of each instalment. A Table can be prepared as shown below:

Amount (Rs)	*Time (months)*
100	10
200	5
250	4
500	2

Here, as the amount of monthly instalment increases, the time of repayment decreases; the ratio of the amount is the multiplicative inverse of the ratio of corresponding time period. For example,

$\frac{100}{200} = \frac{1}{2}$, $\frac{10}{5} = \frac{2}{1}$ which is inverse of $\frac{1}{2}$. Again,
$\frac{200}{500} = \frac{2}{5}$ and $\frac{5}{2}$ is its inverse.

In this case the 100, 200, 10, 5 are said to be in *inverse proportion*.

Continued Proportion

Another interesting type of proportion is the proportion between three terms instead of four. Here the middle term is in proportion with both the first and the third terms. Thus,

$$\frac{a}{b} = \frac{b}{c}$$
$$ac = b^2$$
$$b = \sqrt{ac}$$

Thus, the middle term is the square root of the product of the first and third terms. The middle term is called the *geometric mean* of the other two terms. It is used in many

branches of mathematics and science. It is used as an important statistical average. The idea of a common term in a proportion can be extended to more than three terms. Such a proportion is called a *continued proportion*. For example,

$$\text{If } \frac{a}{b} = \frac{b}{c} = \frac{c}{d} = \frac{d}{e} \text{ etc.}$$

then the quantities a, b, c, d, etc. are said to be in continued proportion.

5

SHAPES AND SIZES

Radhanath Sikdhar was a surveyor working in the office of the Survey of India. He was on his usual duty in his office at Darjeeling. One day he decided to measure the height of a distant peak, that seemed to be the loftiest of all. It was called PK-15. He had already fixed the height of the place where he had established his base. From there he took measurements and made calculations. He could not believe his eyes. He was 29,002 ft (8,700.6 metres) from sea level. This was the highest peak in the world which was later named Mt Everest. He was the first person to find it out. How could he find the height of the peak without actually climbing it? It was no magic. It was his knowledge of geometry that enabled him to achieve this result. Let us see how geometry helps us in dealing with such problems.

'Geometry' literally means, 'measuring the earth' (*geo* means 'earth', *metry* means 'measuring'). However, geometry began primarily as a study of various shapes found in Nature. Look at a few of Nature's beautiful products (Fig. 23).

Each of them shows a certain geometrical shape. For example, the banana leaf shows parallel lines and angles, the beehive shows hexagons, while the quartz crystal shows a six-sided solid figure. Each of these figures shows a particular arrangement of its parts. Such an arrangement is called a 'pattern'. Each of these objects has a different pattern. The fun of it is the fact that we can recognise these

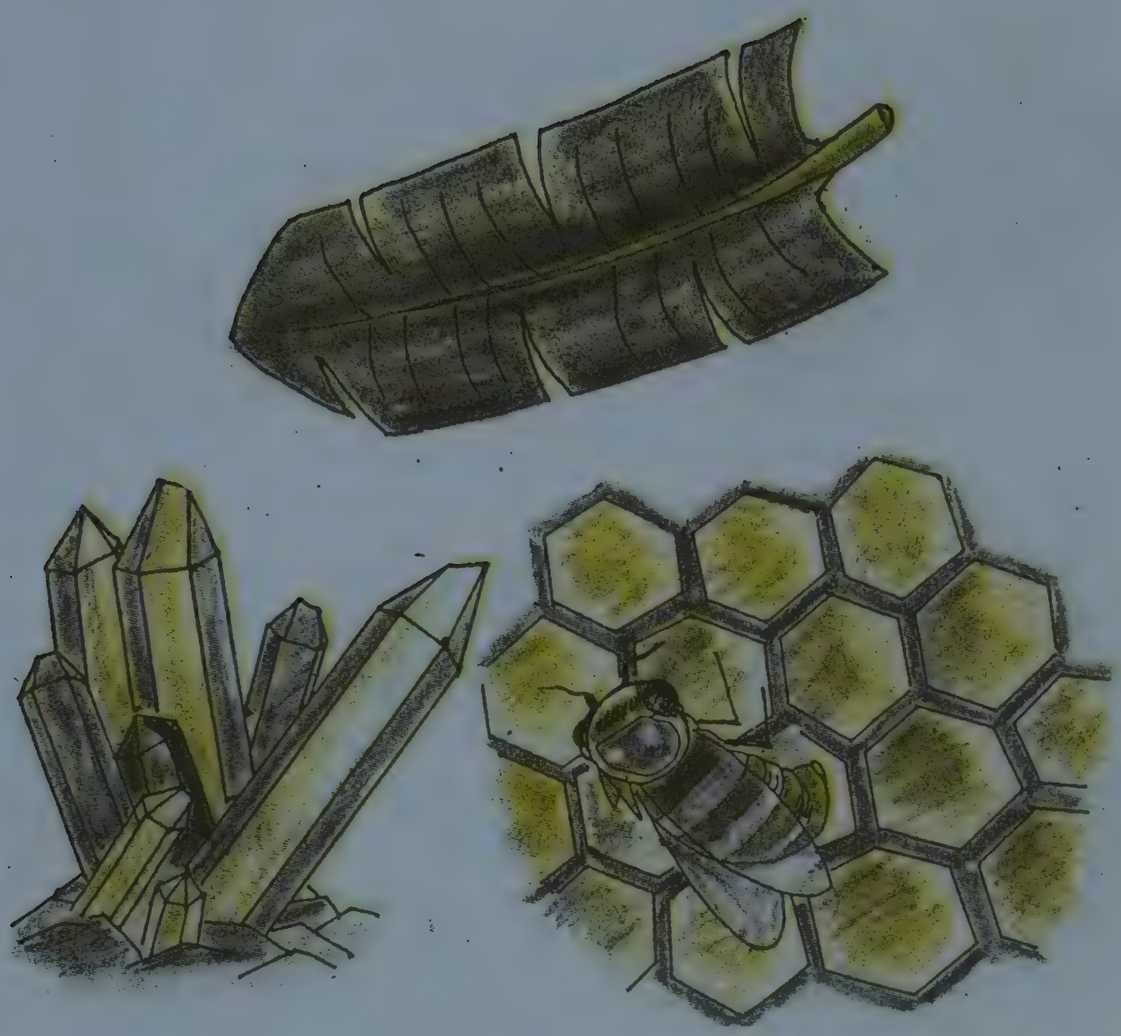

Fig. 23

patterns, can remember them, can understand the difference between them and can recall these patterns if they occur elsewhere.

Development of geometry took shape through the interaction of man with Nature. The bright stars in the night sky gave rise to the concept of a point. The branches of trees, the stars in a configuration gave him the idea of an angle. Beams of light penetrating the gaps in the leaves of trees and made visible by dust or smoke gave rise to the concept of a straight line, while the shapes of leaves gave rise to the idea of a curved line.

The basic shapes of geometry—a point, a line, an angle, and a circle were conceived by observing the patterns of leaves, petals, etc. (Fig. 24). All other shapes, viz. the triangle, quadrilateral (with four sides), hexagon (with

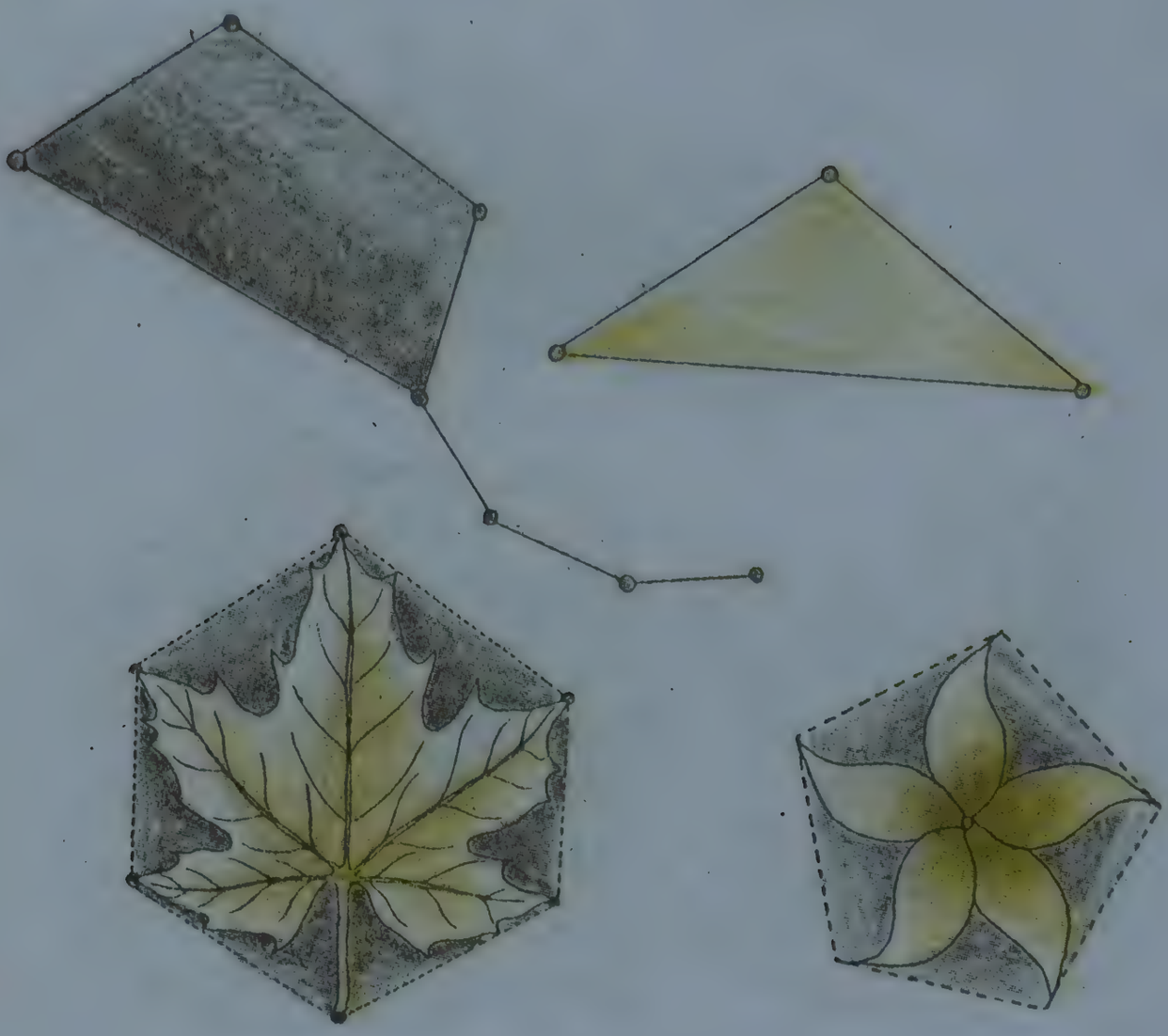

Fig. 24

six sides), etc. were built up of these basic shapes. The Sanskrit names, *trikon* (with three angles), *chatushkon* (with four angles), *panchakon* (with five angles), etc. also exemplify the same thing.

Geometry: A Useful Tool

The beginnings of geometry go back far into prehistory. As the population of a given region grew, the natural dwelling places available did not suffice. It became necessary to build shelters big enough to house families and strong enough to withstand sun, wind, rain and storm. To make a shelter of proper size, a person had to compare lengths and take the height to be higher than the tallest person.

The ancient Babylonians were pioneers in this branch of mathematics. The land between the Tigris and Euphrates

rivers, where the Babylonians dwelt, was originally marshland. Canals were built to drain the marshes, and to catch the overflow of the rivers. For the purpose of canal construction, it was necessary to survey the land. In so doing, the Babylonians developed rules for finding areas. These rules were not exact, but the results they gave were enough for canal construction.

In Egypt, the farmers had farms along the banks of the river Nile which were taxed according to their holdings. In the rainy season, the river would overflow its banks and submerge the land, washing away all landmarks. It became necessary, therefore, to re-measure the land so that each owner would have his rightful share (Fig. 25). After the floods had subsided, trained men or rope-stretchers would establish new landmarks. They would use ropes knotted at equal intervals so that they could measure out desired lengths and divide the land into triangles, rectangles, and trapezoids. They devised practical rules for the areas of these figures which were rough and not exact.

As the civilisations developed all over the world, mathematicians studied the properties of various geometrical figures. It is interesting to note that civilisations in ancient Babylonia (now Iraq), Egypt, India and China, although far

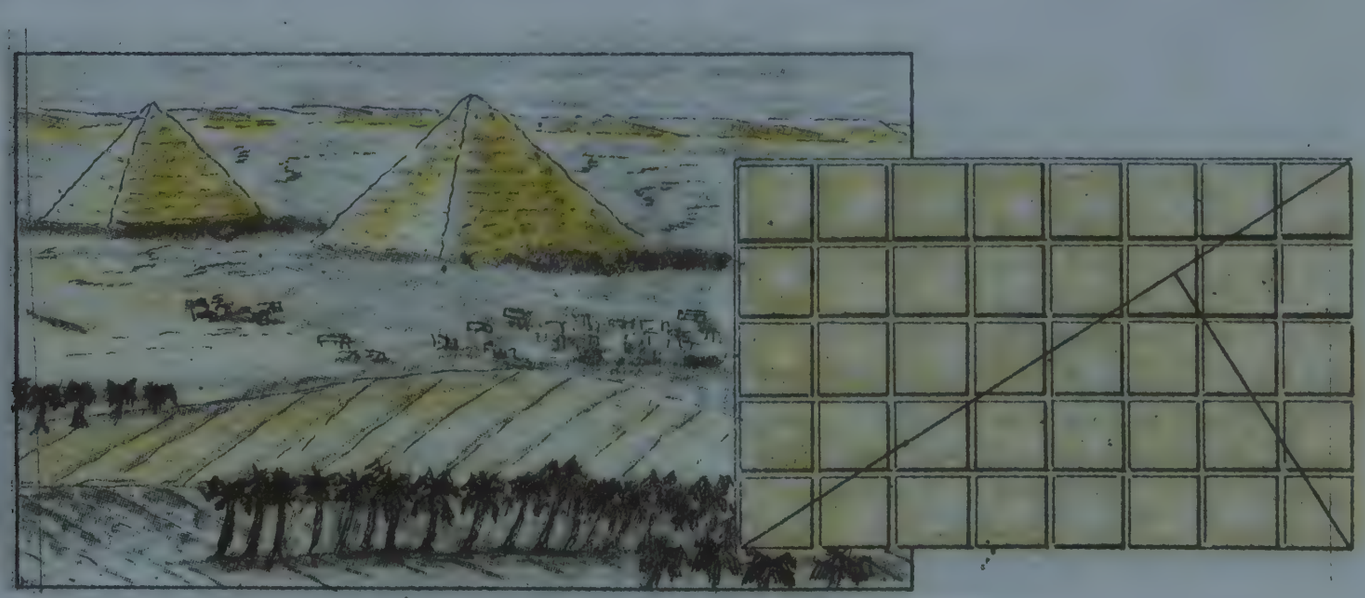

Fig. 25

away from each other had developed almost the same geometry. The people in these countries were great builders, and used the property of a right-angled triangle to draw accurately perpendicular lines, squares and other figures. For instance, the ancient Babylonians and Egyptians knew that if you make a triangle whose sides were 3, 4 and 5 units long, then the angle between the two shorter sides was a right angle (90°). The Egyptians used a rope with knots tied at regular intervals as a ruler, and by using one of these it was fairly easy to construct a 3, 4, 5 triangle to make a right angle (Fig. 26.a). Using such simple tools they could build pyramids and palaces with surprising accuracy. Though today we know that the area of any triangle is one-half the product of its altitude or height, and its base, the Egyptians erroneously gave this area as one-half the product of the base and a side. Since most of the triangles used in their

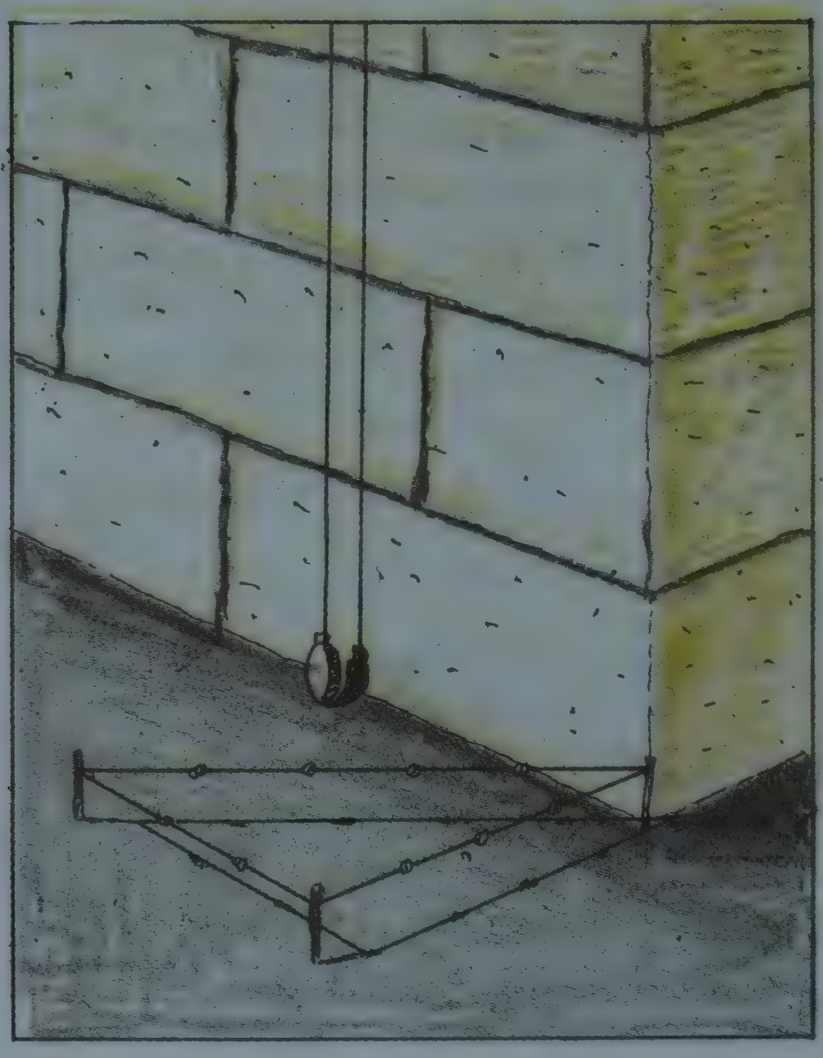

Fig. 26.a

Fig. 26.b

survey work were long and narrow—in such triangles there is not too much difference in length between the side and the altitude—the result of their calculations showed a fairly correct basis for allotting of land and for taxation purposes. The Great Pyramid built in Egypt in 2500 B.C. has a base which is an exact square having sides of length about 230 metres (Fig. 26.b). It was indeed a great feat to draw such a huge square with extremely accurate sides and angles.

In the hands of the Hindus, geometry took an entirely different turn. They developed geometry as a tool to solve practical problems in everyday activities and astronomy. They freely used the numbers and algebra to solve geometrical problems of measuring areas, volumes, and so on. Because of the practise of learning *sutras* by heart, written books

had appeared late in India. The first books of geometry in the form of *Shulba-sutra* are believed to have appeared around 800 B.C. This indeed was much before the Greeks started doing geometry.

In India, the theorem of Pythagoras was known as *Baudhayan Siddhanta* (Theorem of Baudhayan) long before Pythagoras. Baudhayan was a Hindu mathematician who has described the geometry of building altars for religious rites of sacrifice. In ancient India, performing sacrifices was an important part of religion. Altars of various shapes had been prescribed for various sacrifices and had to be built according to the exact specifications.

The people of ancient civilisations had realised the pattern of the movement of the sun, the moon and the stars. They started using it to keep time and their calendars. The science of astronomy told them that every constellation takes about 360 days to appear on the horizon in the east again. Hence, the whole circular distance covered by it was divided into 360 parts and each part was called a *degree*. Thus the measurement of an angle came into being. Each degree was further divided into 60 parts, called *minutes*, and each minute was divided into further 60 parts, called *seconds*. This scale of 60 was the gift of the ancient Babylonians, who had a number system based on 60.

The knowledge of the positions of the constellations helped in regulating their daily life, the occasional social functions and the religious rites. It helped them in fixing the direction of any place. For example, the *Shulba-sutra* tells how to fix the east exactly at the middle of the constellations *chitra* and *swati*, when they are a *yuga* (163.4 cm) above the horizon. More importantly it gave a vigorous impetus to the study of geometry.

Geometry as a Discipline

The Greeks called the early Egyptian surveyors *'geometers'* or 'earth measurers' (in Greek *geo* means 'earth' and *meteria* means 'measurement'). The geometers found out many facts

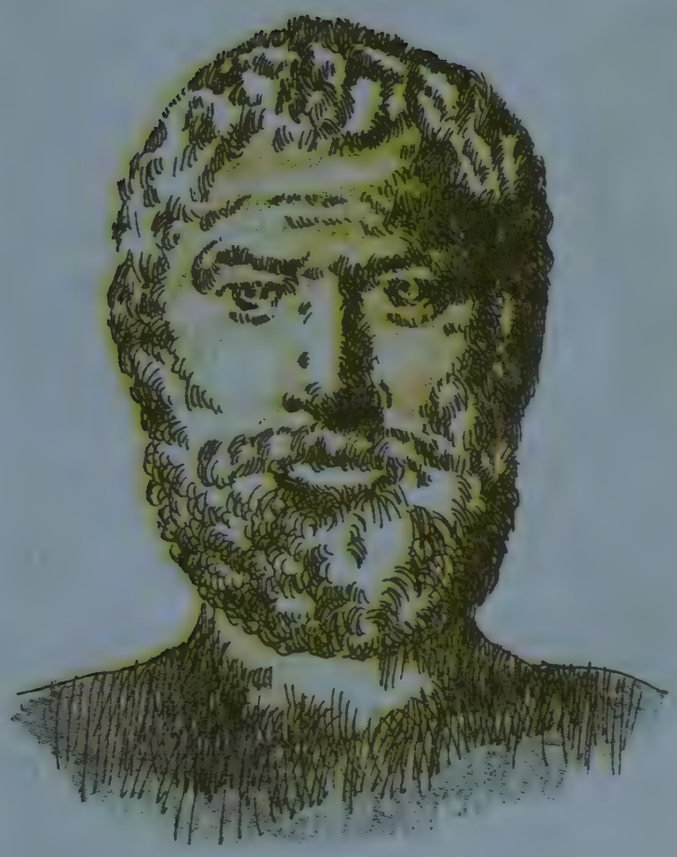

Fig. 27

about triangles, squares, rectangles and even circles. These facts became a body of knowledge that the Greeks called 'geometry' or 'the study of the measurement of the earth'. They made important advances in the field of geometry by not only correcting many of the faulty rules of the Egyptians, but also studying the different geometrical figures in order to work out relationships. Thales, the Greek mathematician (Fig. 27), who lived 2,500 years ago, discovered that no matter what diameter one draws in a circle, it always bisects the circle, i.e. cuts it into two halves. He also noticed that if two straight lines cross each other, the opposite angles are always equal, no matter at what angle the lines cross. This helped the Greeks to change geometry from the study of land measurement to the study of the relations between different parts of the figures existing in space. After Thales, other Greek mathematicians like Pythagoras discovered and proved a number of facts about geometric figures.

By the fourth century B.C., there had grown up a vast body of facts concerning geometric figures like theorems of triangles and circles, but no orderly arrangement. The learned Greek mathematician, Euclid (Fig. 28), who taught at the Alexandria Museum in Egypt, about 300 B.C., was the first man to apply logical development in his *Elements of Geometry*. He used terms, definitions and postulates to prove

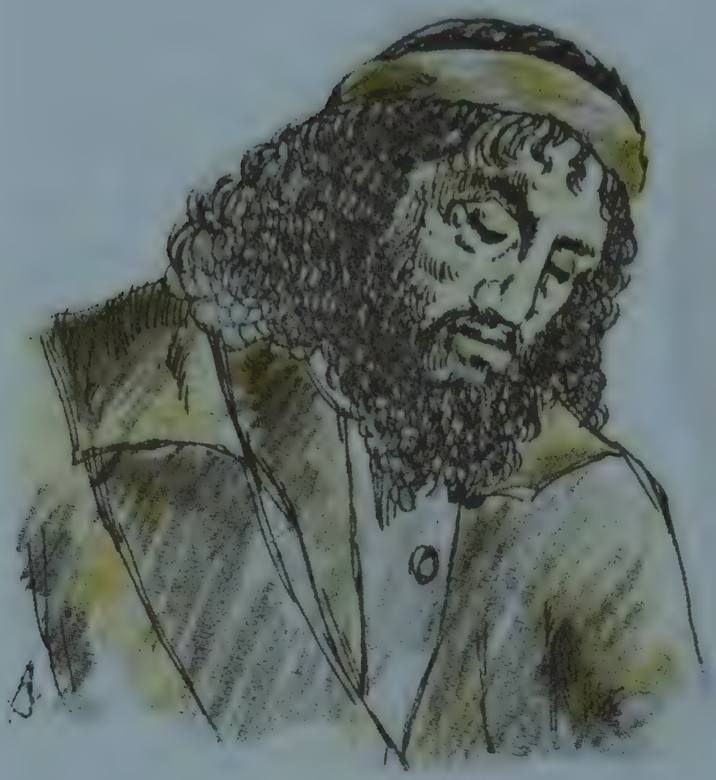

Fig. 28

theorems about geometric figures. Besides, he introduced a method of treatment that became a model for the development of other branches of mathematics. This method is as valid today as ever.

Congruency and Similarity

Of all the principles of geometry, the principles of congruency and similarity are the most important. They form the basis of all our activities. Let us see how.

Congruency

Take a paper. Fold it along a line. Cut any figure from the folded paper, holding the folds together. You get two figures which can be perfectly matched with each other by putting one over the other. Two plane figures are said to be

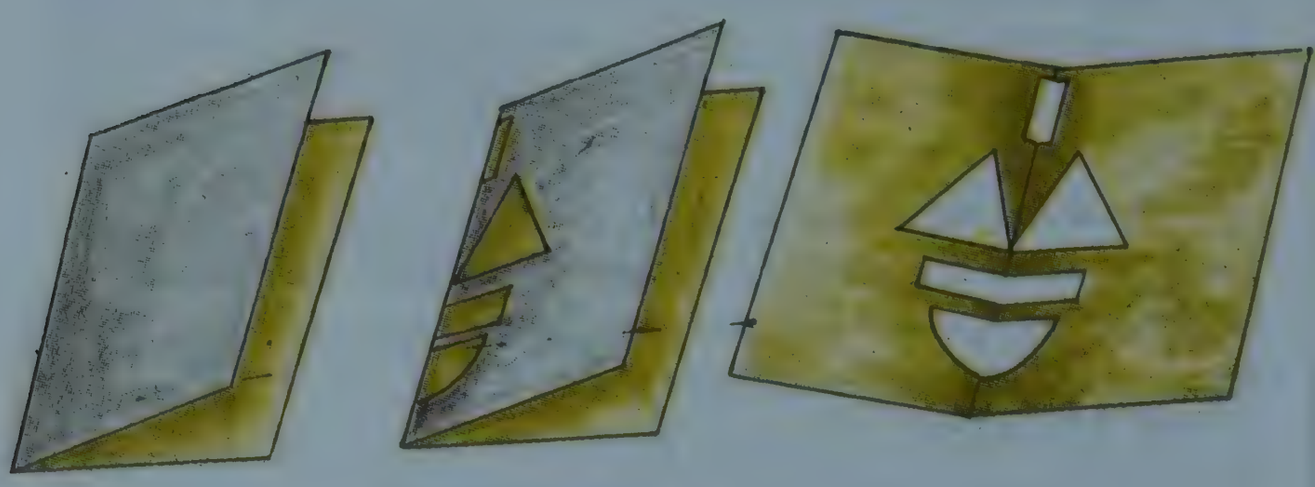

Fig. 29

congruent if they match perfectly when overlapped (Fig. 29). Consider a deck of playing cards. The cards are congruent. They overlap each other perfectly. However, the actual process of overlapping is not necessary to determine the congruency of two figures. Congruency can be established by first establishing a one-to-one correspondence between the elements of the figures and by the measurement of the corresponding elements of the figures. In the case of two triangles if the measures of corresponding sides and corresponding angles are equal then the two triangles are congruent. Thus, two triangles are said to be congruent if for a given correspondence of the vertices, the corresponding sides and corresponding angles are congruent. The congruence of sides and angles is established by the definitions that *segments are congruent if their lengths are equal* and *two angles are congruent if their measures are equal.*

It is an interesting fact that to establish the congruency of two triangles it is sufficient to know the congruence of only three specific corresponding elements of triangles instead of all the six. There are, however, two conditions for this. Firstly, the three elements must contain at least one side, and secondly, if the elements contain only one angle, it must be included by the two sides. For a quadrilateral, the number of conditions goes up to five (i.e. $2 \times 4 - 3$), for a pentagon it is 7 (i.e. $2 \times 5 - 3$), and so on. In general for a polygon of n sides, the number of conditions is $2n - 3$. Two circles are congruent if they have the same radius. Similarly squares having same sides are congruent.

You may ask, "Does congruency have any significance in real life?" The answer is a very emphatic, "Yes!" In fact our real life depends upon our strong belief, although intuitive, in congruency of things. All the things of daily use—cars, radios, TV sets, watches, toys, etc. depend on congruency for their manufacture. To manufacture a watch, for example, dies are prepared to produce various parts of it. Each part produced from a die is exactly like the others in all respects. One part of a machine can be replaced by another

part only because we know that they are exactly alike, i.e. congruent. All the pages of a book are congruent and hence can be properly compiled and bound. If we want to measure length, we use any ruler that is available because we know that all rulers are made congruent.

Similarity

Look at the figures. The triangles are equilateral (Fig. 30), the quadrilaterals are squares, the hexagons are regular. The corresponding angles are congruent. The sides are parallel

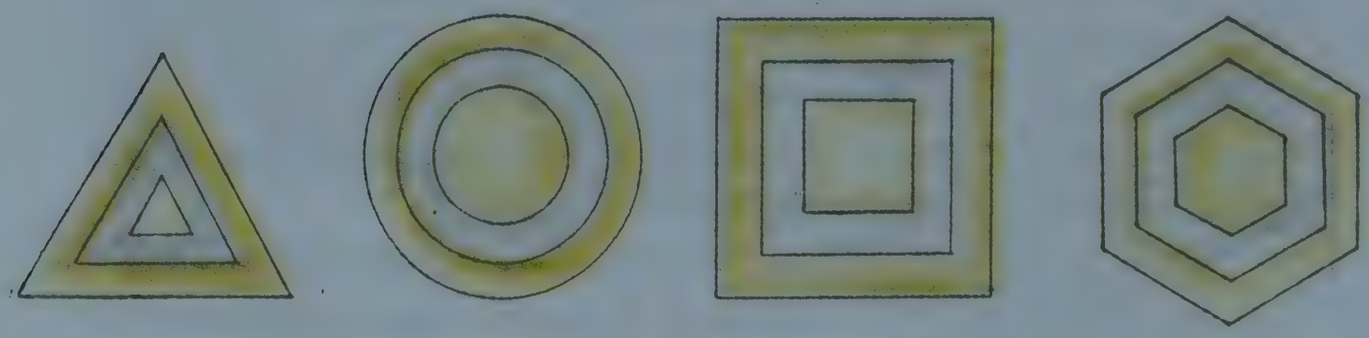

Fig. 30

but not congruent. The figures in such a case look alike. They are called similar figures. The necessary conditions for figures to be similar are that the corresponding angles must be congruent, and secondly, the ratios of the lengths of corresponding sides should be equal. All circles are, of course, similar.

Look at the portrait in Fig. 31. The two sketches show the same person but in different sizes. Each point in one figure corresponds to exactly one point in the corresponding figure. However, the distance between two points in one is not equal to that between the corresponding points in the other. They are 'scaled down' in the same proportion in the smaller figure. In other words, these are similar figures.

Nature, too, abounds with similar shapes. The best example is provided by crystals that occur in Nature. All crystals of a crystalline substance are exactly alike in shape

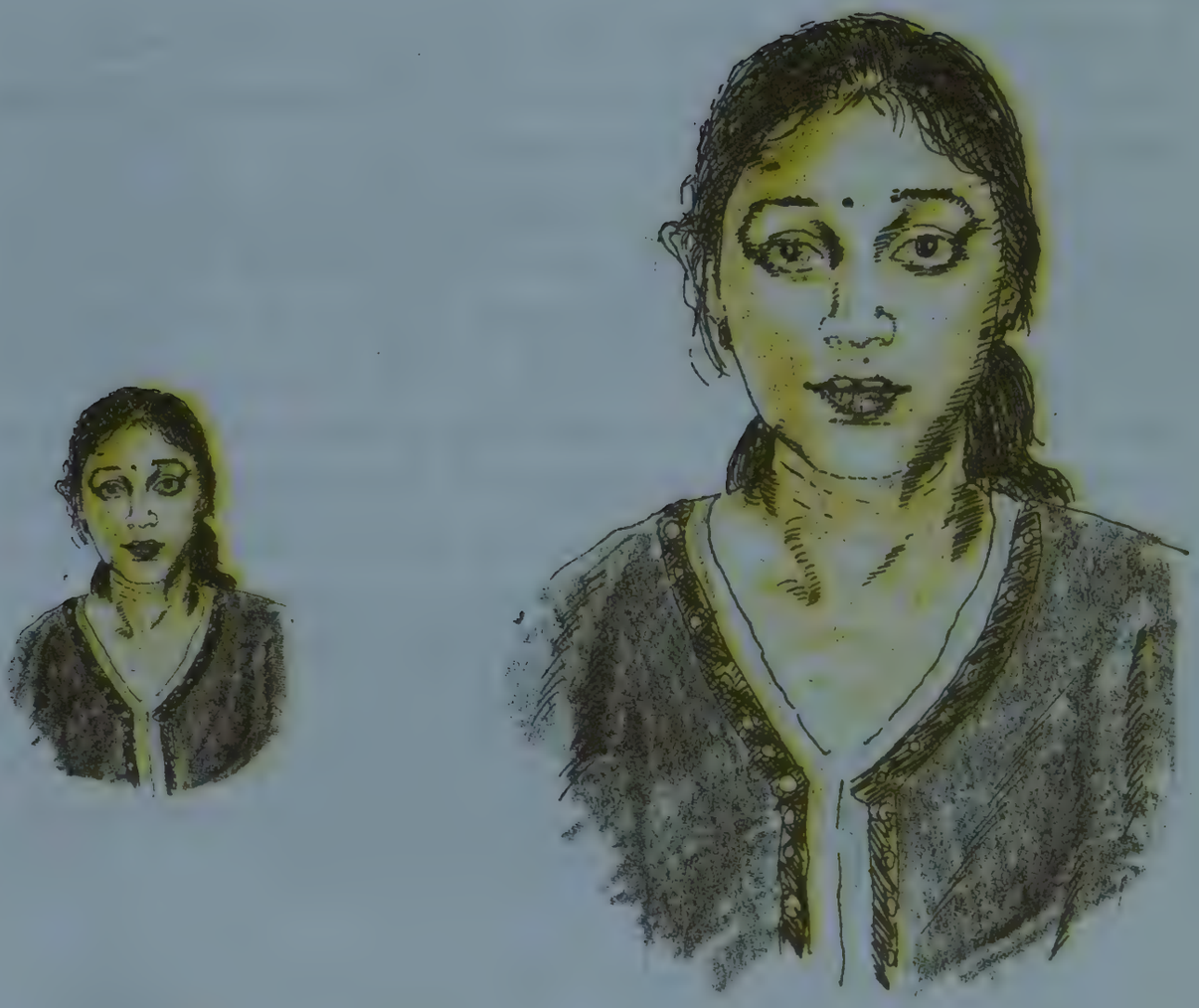

Fig. 31

though different in size. If a crystal breaks into smaller crystals, it always breaks into similar crystals. Soap bubbles of various sizes, small and large flowers of the same kind, the hexagonal holes of a honeycomb, beautiful designs on wings of a butterfly are the common examples of similarity occurring in Nature.

Why are soap bubbles always spherical? Why are droplets of water always spherical? Has it anything to do with geometry? Yes, certainly. Nature tries in each case, to acquire a surface of minimum area, to achieve the state of least surface energy. The sphere is the shape of a solid which has a surface area smaller than any other solid of equal volume. Hence, soap bubbles or liquid droplets always acquire a spherical surface. For the same reason the planets also are spherical.

Symmetry

Is symmetry useful in life? The question should really, be turned around. Is life possible without symmetry? Let us answer this question in a little detail.

Look at Fig. 32. It shows various arrangements of leaves on a branch or the petals of a flower. The various shapes like triangle, square, pentagon, hexagon or a circle are quite familiar to us. In fact, it is Nature which has suggested all these shapes to man. In all these shapes one can find almost perfect symmetry in the arrangement. Symmetry is nothing but the congruence of the parts of a figure with respect to a line or a point.

Consider the Fig. 32.a. The figure is symmetric about the vertical line passing through the centre of the figure and the vertex of the leaf. The figure would be identical if we turn it through an angle of measure 120°. Thus the figure

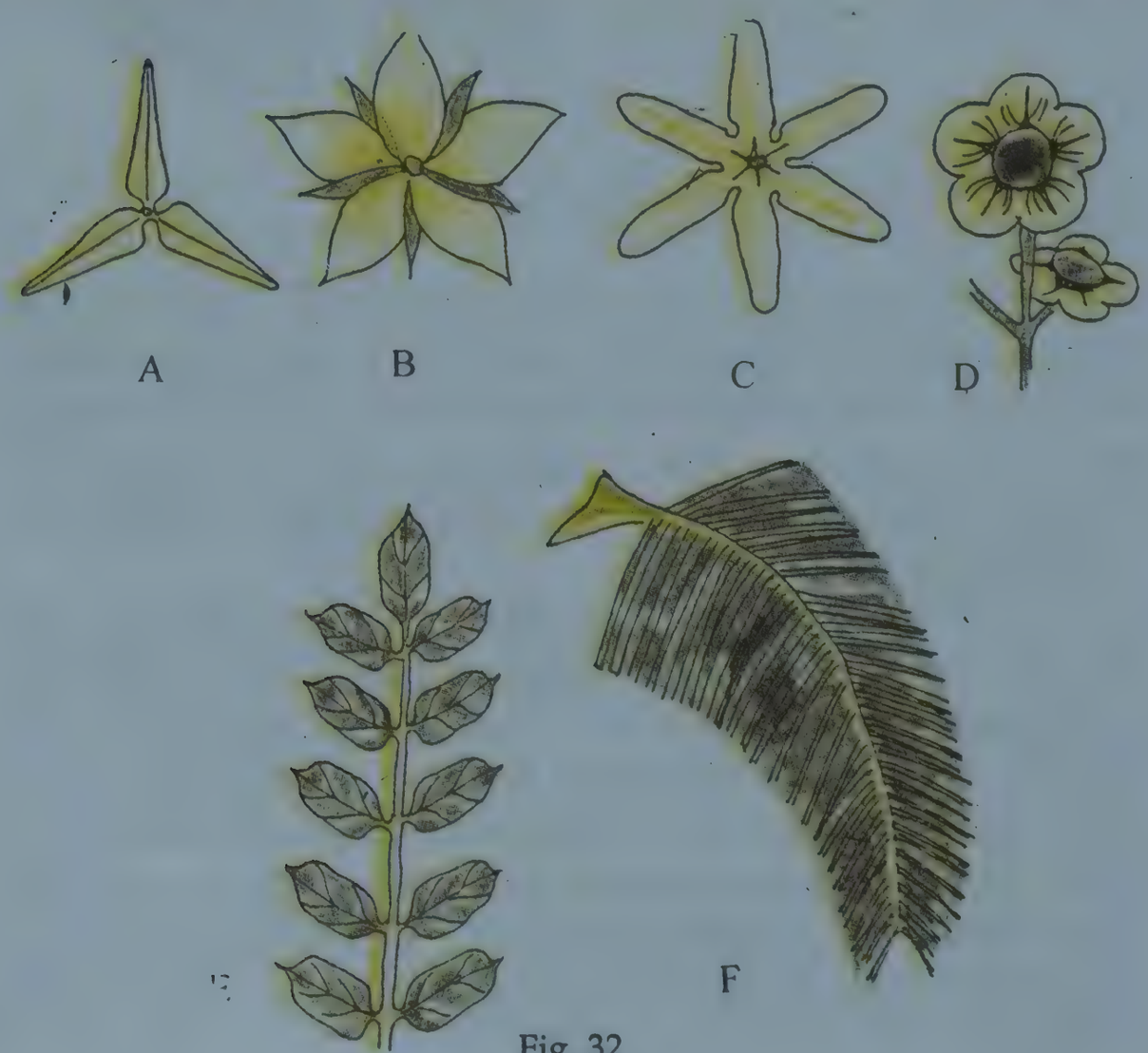

Fig. 32

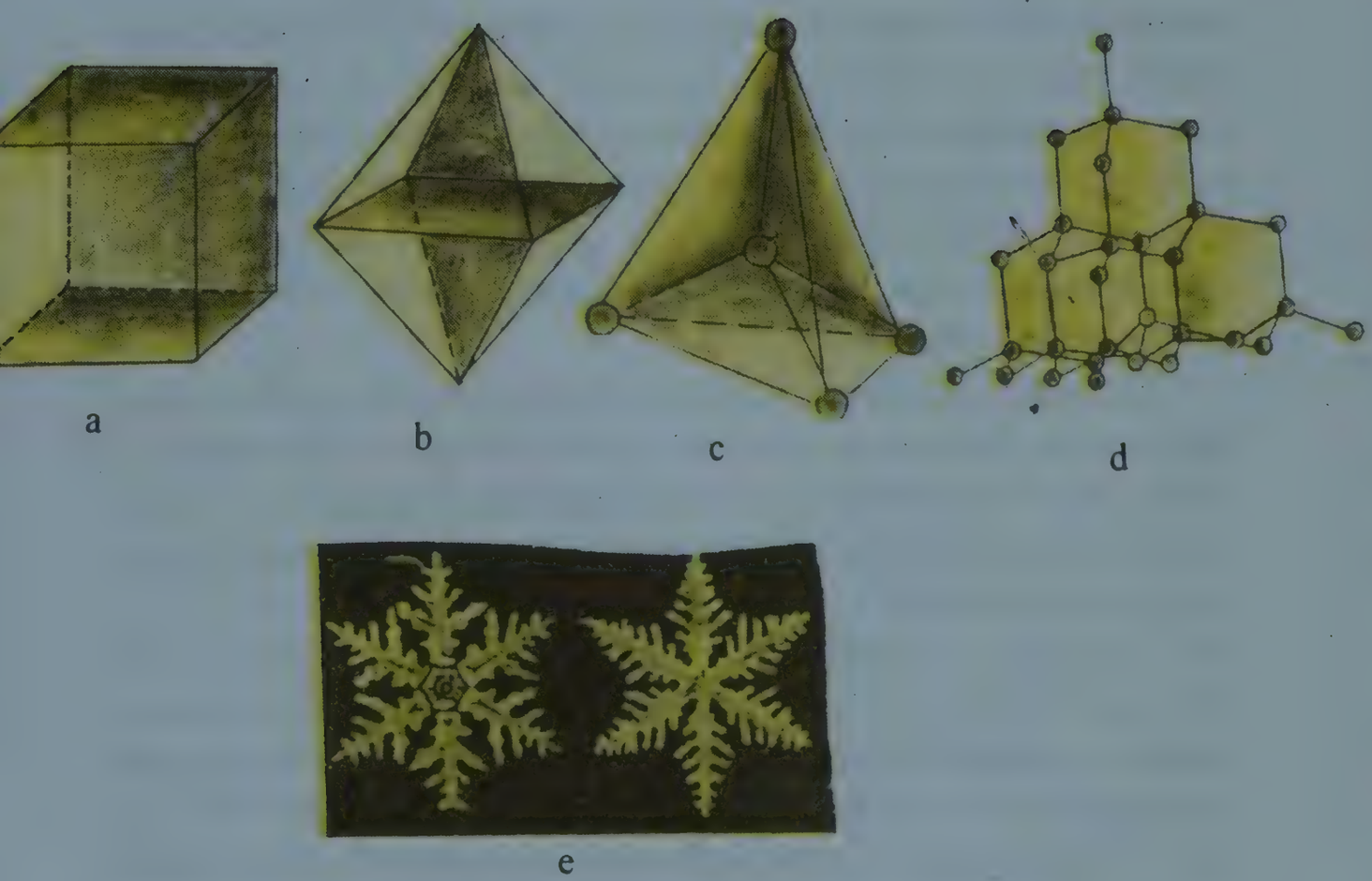

Fig..33

has an axial as well as rotational symmetry. All the figures shown here, except 32.e and 32.f, have both axial and rotational symmetries. The leaves show only axial symmetry.

The symmetry exists in many other natural objects. Consider the crystals of common salt. They are cubic (Fig. 33.a). The cube is an object having many axes of symmetry. The crystals of rhombic sulphur show another example of a symmetrical crystal (Fig. 33.b).

A tetrahedron is another symmetrical shape favoured by Nature. The most precious of all the gems, diamond, is made up of a network of tetrahedrons formed by carbon atoms. Quartz is another example of tetrahedral network of crystals (Fig. 33.e).

Hexagon is another symmetrical shape favoured by Nature. While the tetrahedral network of carbon atoms makes diamond the hardest of all substances, the hexagonal arrangement of carbon atoms in a plane imparts smoothness to graphite crystals and makes graphite an ideal lubricant for heavy duty machines (Fig. 33.f). The hexagonal shape of a hole in a honeycomb makes it the best space-filling shape.

Perhaps the most important geometrical shape favoured by Nature is the helix, though it is not perfectly symmetrical. Helix is a curve which winds itself around a cylinder. A screw is a familiar example of a helix. In Nature we can see a helix in the form of a creeper which climbs a tree or a tendril winding itself in the form of a spring (Fig. 33.g). What is not easily seen is the molecule of DNA in the nucleus of a cell, which is made up of a double helix. Each strand of the double helix is made up of a long chain of nucleotides which contain a sugar molecule, a phosphate molecule and a molecule of a nitrogen base.

Geometry at Home

One can say that our lives are governed by rectangles and circles. Think of the books, tables, cupboards, walls, floors and the roofs, the coins and cups and dishes and what not.

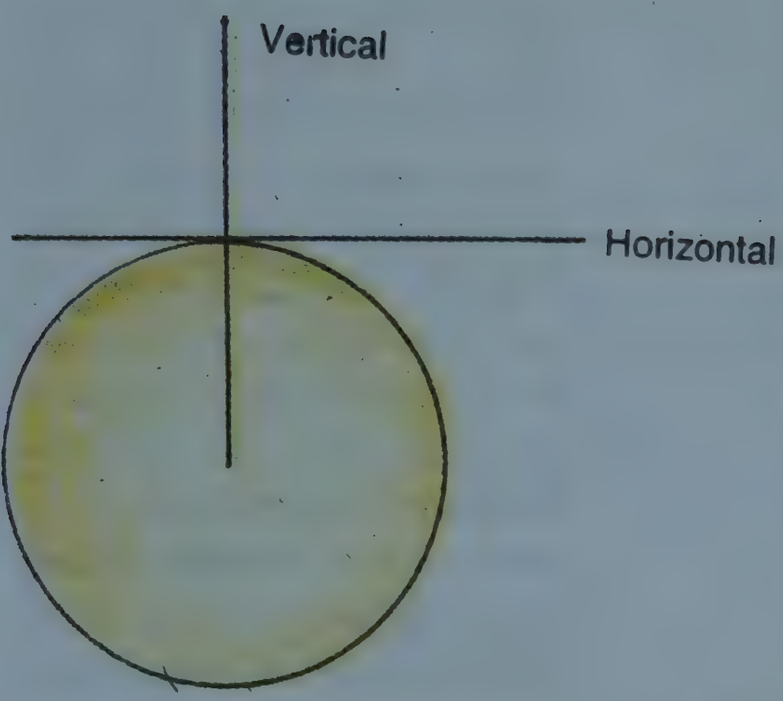

Fig. 34

Call a carpenter and he would be there with his setsquare and ruler. Call a mason and he would come with his spirit level and the plumbline. Why do they need these setsquares and plumblines? The reason is simple. The gravity of the earth and the spherical shape of the earth force us to use only two directions, the horizontal and the vertical. The force of gravity pulls everything towards the earth's centre, along its radius. The tangent plane to the surface of the earth at any point is at right angles to the radius of the earth through that point. The plane is called *horizontal* while the line along the radius is called the *vertical* (Fig. 34). If we put anything on the ground, it will be stable only if the surface of the ground is exactly horizontal. Keep a ball on a surface which is not horizontal. It rolls down the slope. Pour water on it. It flows down the slope. If the pillars and walls of a building are not built exactly vertical the building would be unstable. Thus Nature itself forces us to use the horizontal and the vertical directions.

A rectangle is the simplest shape to form and use. It is a very convenient shape to fill the space. The setsquare of a carpenter is a right angle, set up permanently, which enables to form perfect right angles at the corners of a rectangle or a square without difficulty. It also enables him to draw a right angle at any point on a line and also to draw parallel lines (Fig. 35).

When a carpenter fits the planks in a cupboard, he ensures that the perpendicular distances AC and BD are equal,

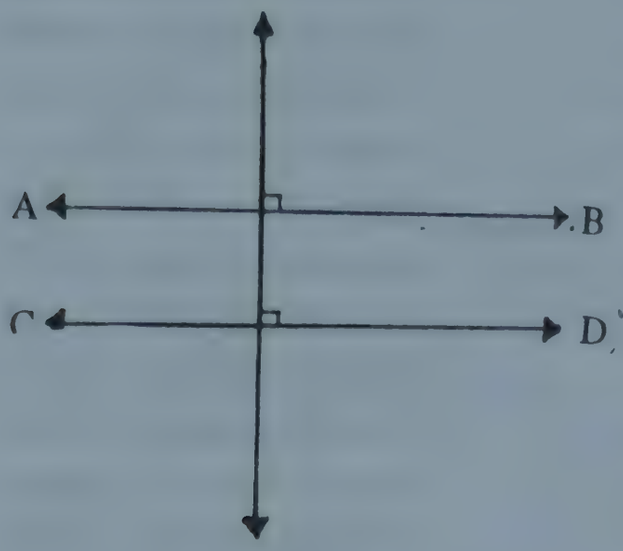

Fig. 35

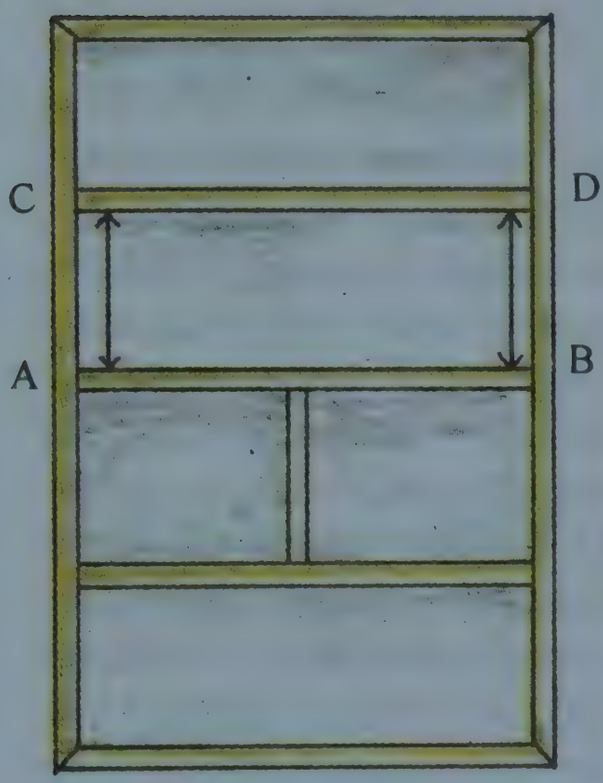

Fig. 36

so that the plank *AB* is parallel to the plank *CD*. The planks must be parallel and horizontal to keep the things exactly balanced (Fig. 36). The sprit level and the plumbline of a mason achieve exactly the same purpose, that is, deciding the true horizontal and the vertical. Free surface of any liquid is always horizontal. Thus the bubble of air in the spirit level shows whether any line or plane is horizontal or not. The plumbline when suspended freely is always vertical (*see* Fig. 26.a on p. 54).

Measuring areas and volumes is again a daily activity completely based on geometrical formulae. The carpenter, the mason, the painter and a worker digging the earth and making a heap of stones or soil, all measure their work in areas and volumes.

Geometry can also provide answers to our questions to satisfy our curiosity. Suppose a man 2 metres tall stands on the ground, how far is the horizon from him?

Look at Fig. 37. It is not drawn to scale. However, it serves our purpose. Let *AB* be the height of the man. Thus $AB = 2$ metres. Let *BC* be the diameter of the earth. Hence $BC = 13400 \times 10^3$ metres. Let *D* and *E* be the points on the

horizon. Then we have to find the distance BD or BE where AD and AE are the tangents. Now by a theorem on circles we have

$$\begin{aligned}(AD)^2 &= (AB) \times (AC)\\ (AD)^2 &= 2 \times 13400 \times 10^3\\ &= 268 \times 10^5 \text{ metres}\\ (AD) &= \sqrt{26.8 \times 10^6} = \sqrt{26.8 \times 10^3} \text{ metres}\\ &= 5.176 \times 10^3 \text{ metres}\\ &= 5.176 \text{ km.}\end{aligned}$$

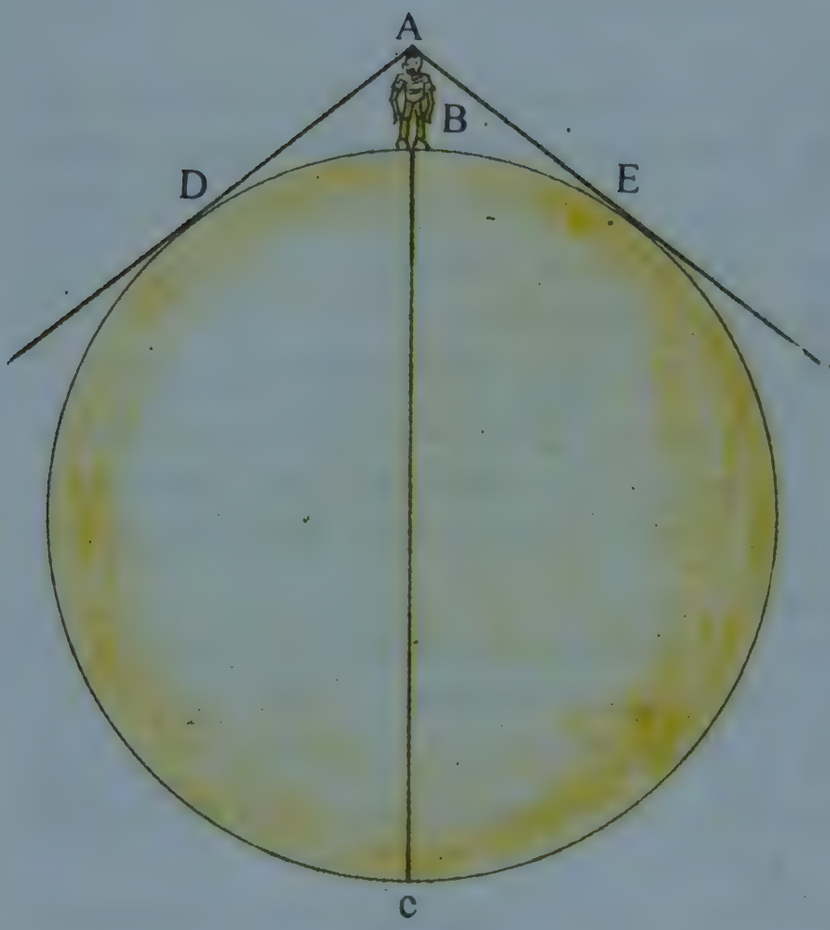

Fig. 37

Now the point D is so far away that the distance BD and distance AD are practically equal. Hence we get $BD = 5.176$ km. Thus for a man 2 metres tall, the horizon is at a distance of 5.176 km.

This simple curiosity of a common man becomes a very important problem of communication with thousands of people when it comes to decide the height of a TV tower. How high a TV tower should be to telecast its programmes up to a certain distance around the TV station? For example, the horizon of a TV tower 200 metres high would be 5.176×10^4 metres, i.e. 51.76 km away. Thus the programmes telecast from this tower would reach the distance of about 52 km.

6

TRIGONOMETRY

An important offshoot of geometry is trigonometry, or triangle measurement. In trigonometry where certain parts of triangles are known, one can determine the remaining parts and thus solve a variety of problems.

Consider the following situations:

- A beautiful palm tree stands majestically in a garden. While enjoying the beauty of this palm every spectator wonders about its height. Can its height be measured without taking the trouble of actually climbing it all the way to the top?
- A person standing on a bank of a wide river, wants to know how wide the river is, without crossing.
- While surveying a difficult terrain, a surveyor wants to fix the distance of an inaccessible tree from where he is standing.

All the situations are quite common and answers could be found easily by using simple properties of triangles. A special branch of mathematics has been developed which studies the relations between the sides and angles of a triangle. This branch is known as *trigonometry*. Let us see how trigonometry helps us to solve the above problems.

Take the case of a simple triangle ABC, with one of its angles a right angle (Fig. 38). Let angle $B = 90°$, angle $A = 30°$ and angle $C = 60°$. Let a, b, c be the lengths of its sides as shown. Now by a well-known property of this triangle,

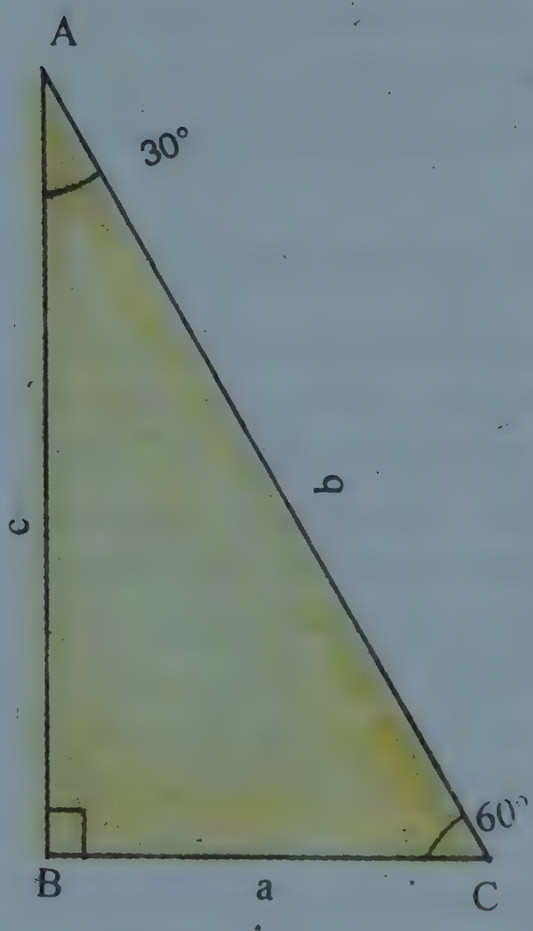

Fig. 38

we get $a = \frac{b}{2}$, and using the Pythagoras theorem we can deduce that $c^2 = b^2 - a^2$, i.e. $c^2 = b^2 - \frac{b^2}{4} = \frac{3}{4} b^2$. Thus $c = \sqrt{\frac{3b}{2}}$. Thus the three sides of this triangle are b, $\frac{b}{2}$ and $\sqrt{\frac{3b}{2}}$. Hence, if we take the ratio of the three sides of triangle ABC, we get $\frac{a}{b} = \frac{1}{2}$, $\frac{c}{b} = \sqrt{\frac{3}{2}}$ and $\frac{c}{a} = \sqrt{3}$. We can see here that each of these ratios has a fixed value. These ratios are given special names with respect to the angle in consideration. If angle C, whose measure is 60°, is referred to, the names of three ratios are as follows: $\frac{a}{b}$ is called cosine of angle C, $\frac{c}{b}$ is called sine of angle C and the ratio $\frac{c}{a}$ is called tangent of angle C. For convenience these ratios are denoted by cos, sin and tan respectively. Thus,

$$\cos 60° = \frac{1}{2},\ \sin 60° = \sqrt{\frac{3}{2}},\ \tan 60° = \sqrt{3}$$

The most important point here is that the values of these ratios depend only on the angles and not on the lengths of the sides. Thus in Fig. 39 the value of $\cos 60° = \frac{1}{2}$ in both the triangles. The values of these ratios are calculated for all the angles from 0° to 90° and Tables of these values are

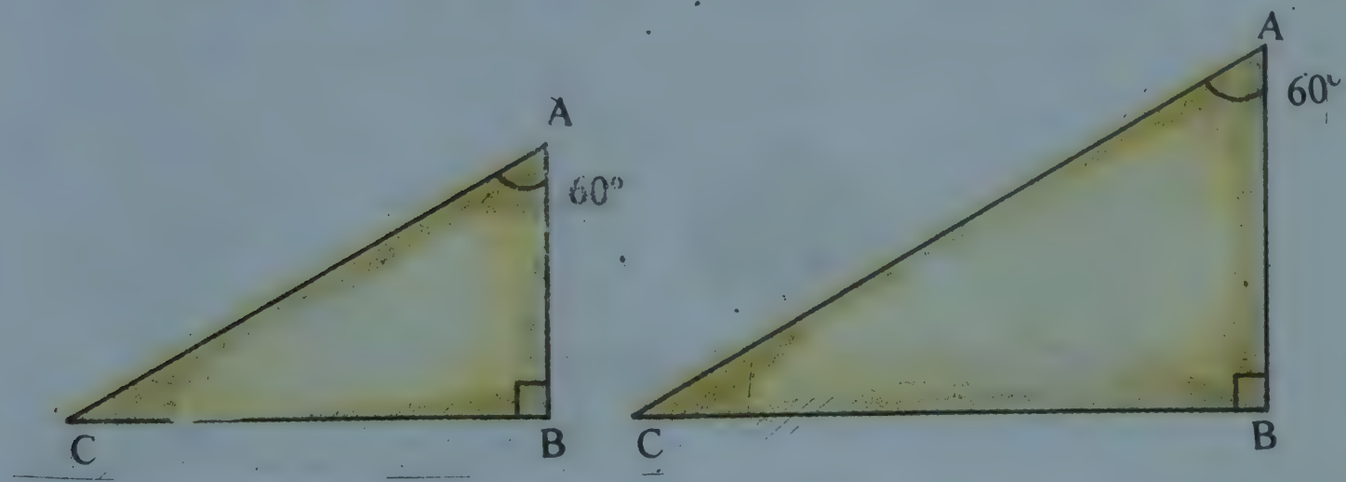

Fig. 39

compiled. These Tables are called *trigonometric tables*. By referring to these Tables we can find the value of any ratio we want. Thus referring to Fig. 39, we get from the Tables the following values

$$\cos 60° = 0.5000, \sin 60° = 0.8660, \tan 60° = 1.7321$$

Preparing accurate trigonometric Tables was an important task which had occupied the brains of the best mathematicians of ancient times. Hipparchus of Nicaea, a Greek mathematician (150 B.C.), was possibly the first to prepare these Tables. It was he who attempted to measure the size of the sun and the moon and their distances from the earth. He felt the need for the type of mathematics that, by applying measurements from the earth, would enable him to measure the objects in space. He was responsible for the invention of trigonometry.

How does trigonometry help us in solving problems? Well, by using a method known as *triangulation* where we measure the angles in a number of triangles and use the trigonometric Tables. Let us represent the situations in simple diagrams.

To solve the first problem, let the tree be shown by the line AB (Fig. 40). BC is shown along the ground and angle $ABC = 90°$. Let $BC = 10$ metres. For an observer at C, when he looks up at the top of the tree, the line of sight makes an angle of 60° with the ground. Here $\frac{AB}{BC} = \tan 60°$. Thus,

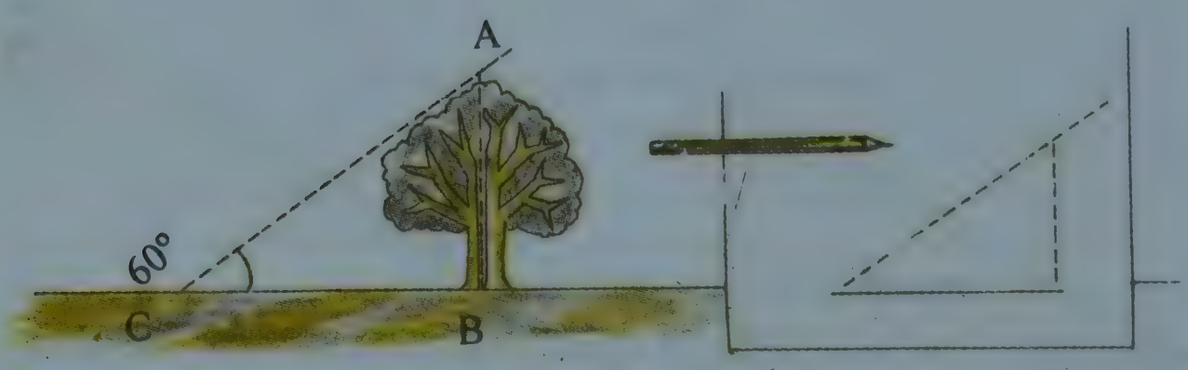

Fig. 40

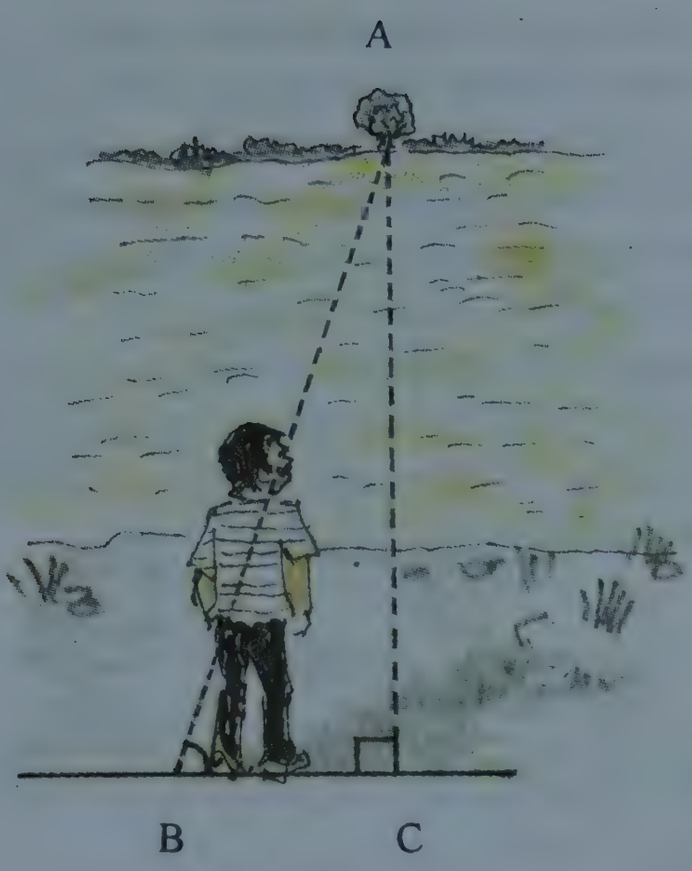

Fig. 41

$\frac{AB}{BC}$ = 1.7321 metres.

Thus, $AB = 1.731 \times 10 = 17.321$ metres, or the height of the tree is 17.321 metres. The use of trigonometry has reduced the problem to two simple measurements, namely, the distance BC along the ground and the angle C.

In the second problem, suppose the observer is standing at the point B on one bank of the river (Fig. 41). He chooses a point A, say, a tree on the other bank of the river. The angle ABC is measured and is found to be 72°. The observer now walks a distance of 30 metres along the river bank up to a point C directly opposite point A on the opposite bank, so that the angle ACB is 90°. Now from triangle ABC, we get $\frac{AC}{BC} = \tan 72°$, $AC = 30 \times \tan 72° = 30 \times 3.077 = 92.310$. Thus the width of the river is 92.310 metres.

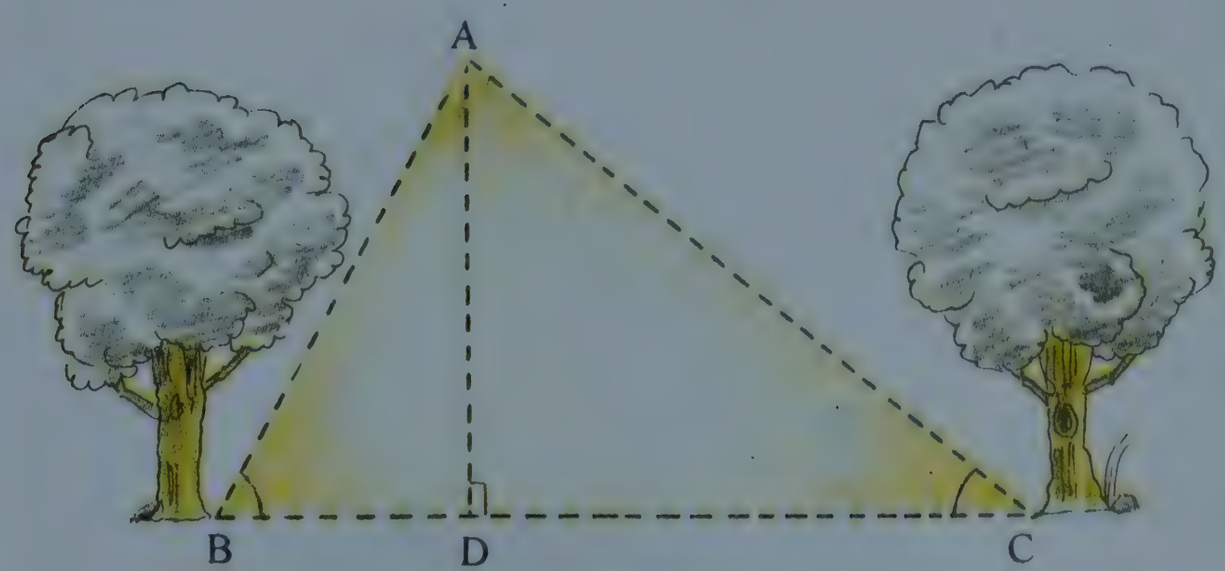

Fig. 42

In the third problem, suppose the surveyor has already measured the distance between the two points B and C, denoted by, say, two big trees (Fig. 42). He wants to find the distance AB, AC and AD. However, point A is inaccessible and hence he cannot measure these distances directly. He can however measure the angles ABC and ACB.

Now, suppose that the distance BC is 100 metres, angle $ABC = 60°$ and angle $ACB = 50°$. From the trigonometric Tables he finds that $\tan 60° = 1.7321$ and $\tan 50° = 1.1918$. He can find the distance AD from the relation

$$AD = \frac{100 (\tan 60° \times \tan 50°)}{(\tan 60° - \tan 50°)}$$

$$AD = \frac{100 \times 1.7321 \times 1.1918}{1.7321 - 1.1918} = 70.60 \text{ metres.}$$

Now distances AB and AC can be found from the relation

$$AB = \frac{AD}{\sin 60°}$$

$$\text{and } AC = \frac{AD}{\sin 50°}$$

$$AB = \frac{70.60}{0.8660} = 81.52 \text{ metres}$$

$$\text{and } AC = \frac{70.60}{0.7660} = 92.16 \text{ metres.}$$

Many other problems of similar nature can be dealt with by using trigonometric formulae and Tables. For example, the height of a distant tower or a building or a cliff

Fig 43

can be found out even without going near the place. The situation is illustrated in Fig. 43.

C is the top of a cliff across a creek. An observer standing at A on the shore in front of the cliff, measures angle CAB to be 55°. He walks a distance of 100 metres to B and measures the angle CBD to be 65°. From the relations $\frac{h}{BD} = \tan 65°$ and $\frac{h}{100 + BD} = \tan 55°$, the height h can be obtained by the relation

$$h = \frac{100 \times \tan 55°}{\tan 65° - \tan 55°} \times \tan 65°$$

$$= \frac{100 \times 1.4281 \times 2.1445}{2.1445 - 1.4281}$$

$$= 427.5 \text{ metres.}$$

Thus the height of the cliff is found to be 427.5 metres. The most spectacular example of the use of trigonometry was the computation of the height of the highest mountain in the world, Mt Everest.

References

1. Graham Flagg: *Numbers*, Penguin Books.
2. James R. Newman: *The World of Mathematics*, Vol. II, Tempus Books.
3. Lancelot Hogben: *Mathematics for the Millions*, Pocket Books.
4. —: *Mathematics in the Modern World*, The Scientific American Publication.
5. Morris Kline: *Mathematical Thought*, Vol. I, Oxford University Press.
6. N. Datta and Singh: *History of Hindu Mathematics*.
7. N.H. Phadke: *Lilavati Punar Darshan* (Marathi).
8. R.P. Kulkarni: *Four Shulba-sutras* (Marathi), Sahitya Sanskriti Mandal, Maharashtra.

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